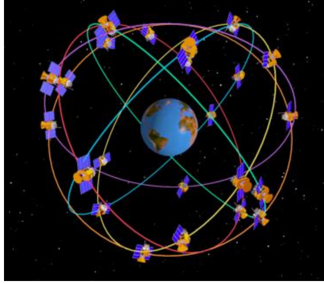
Relativity in Global Satellite Navigation Systems



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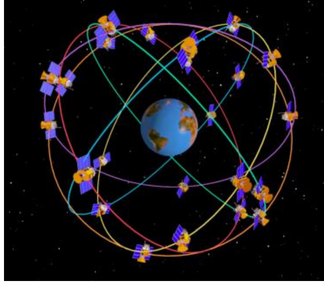
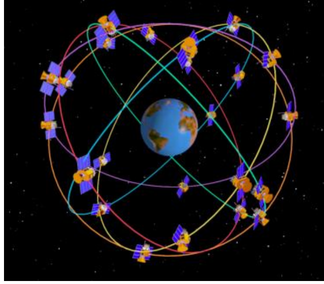
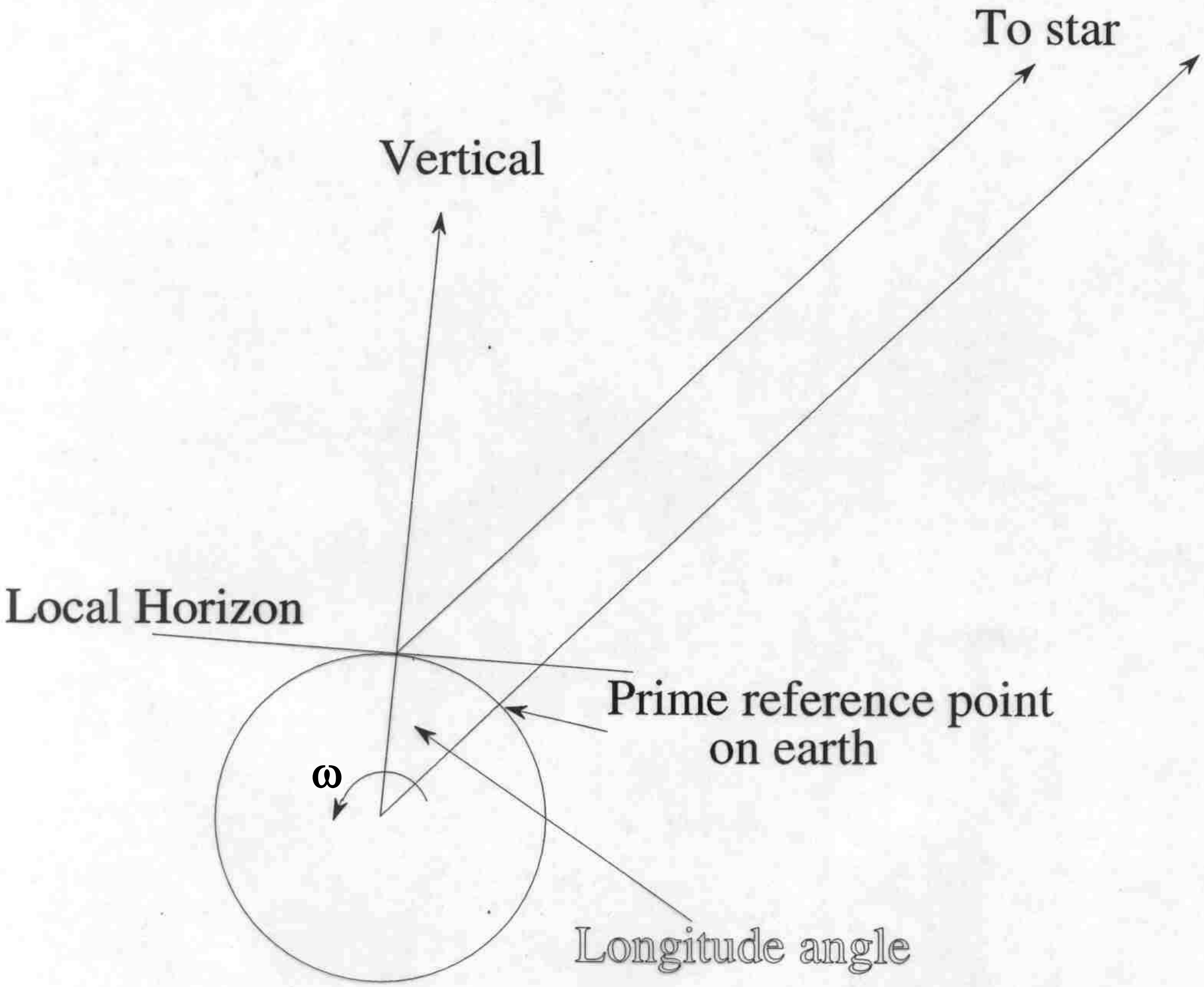
1. Navigation—why you need a clock
2. Brief history of relativity in the GPS
3. What the GPS is
4. Relativistic effects:
   - Relativity of synchronization;
   - Time dilation;
   - Gravitational frequency shifts;
   - Sagnac effect;
5. Observations: testing relativity
   - TOPEX;
   - Frequency jumps;
   - Unmodeled effects;
6. Applications

Latitude

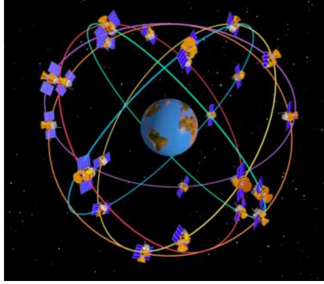




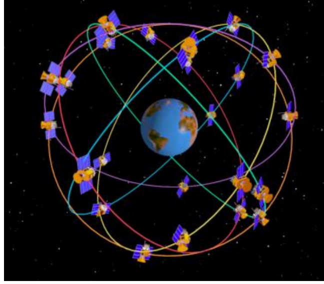
NAVIGATION WITH A CLOCK



Moon, Jupiter & Satellites

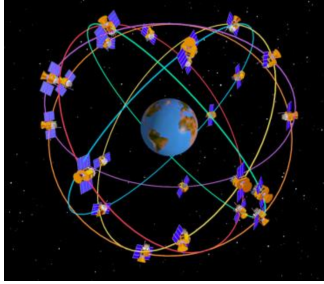


GPS RELATIVITY MEETINGS



• 1979	SAMSO Relativity Seminar (Boulder)
• 1985	JASON Study
• 1986	Air Force Studies Board
• 1988-98	Various Working Group Meetings
• 1995	ARL-Chapel Hill
• 1997	ICD-200 Relativity Review (Boulder)

GPS RELATIVITY MEETINGS

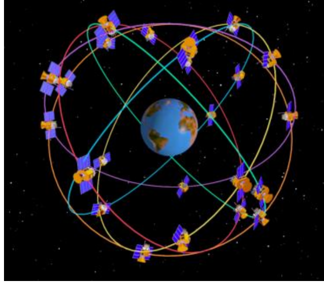


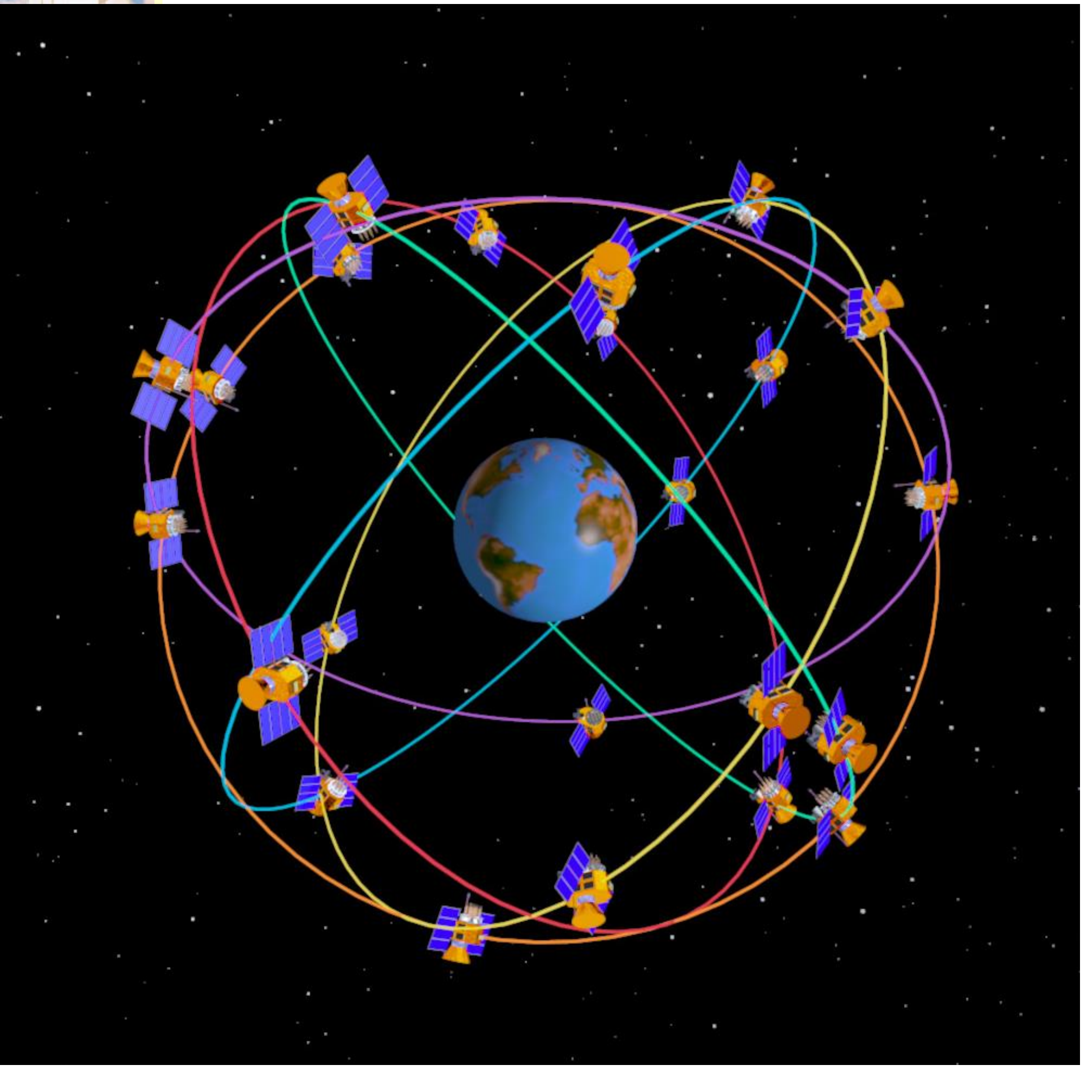
- 1979 SAMSO Relativity Seminar
- 1985 JASON Study
- 1986 Air Force Studies Board
- 1988-98 Various Working Group Meetings
- 1995 ARL-Chapel Hill
- 1997 ICD-200 Boulder

Erroneous Reports

- 1977-83
- 1992
- 1996
- 2000-2006
- 2008

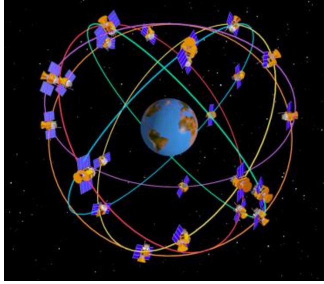
GPS Constellation

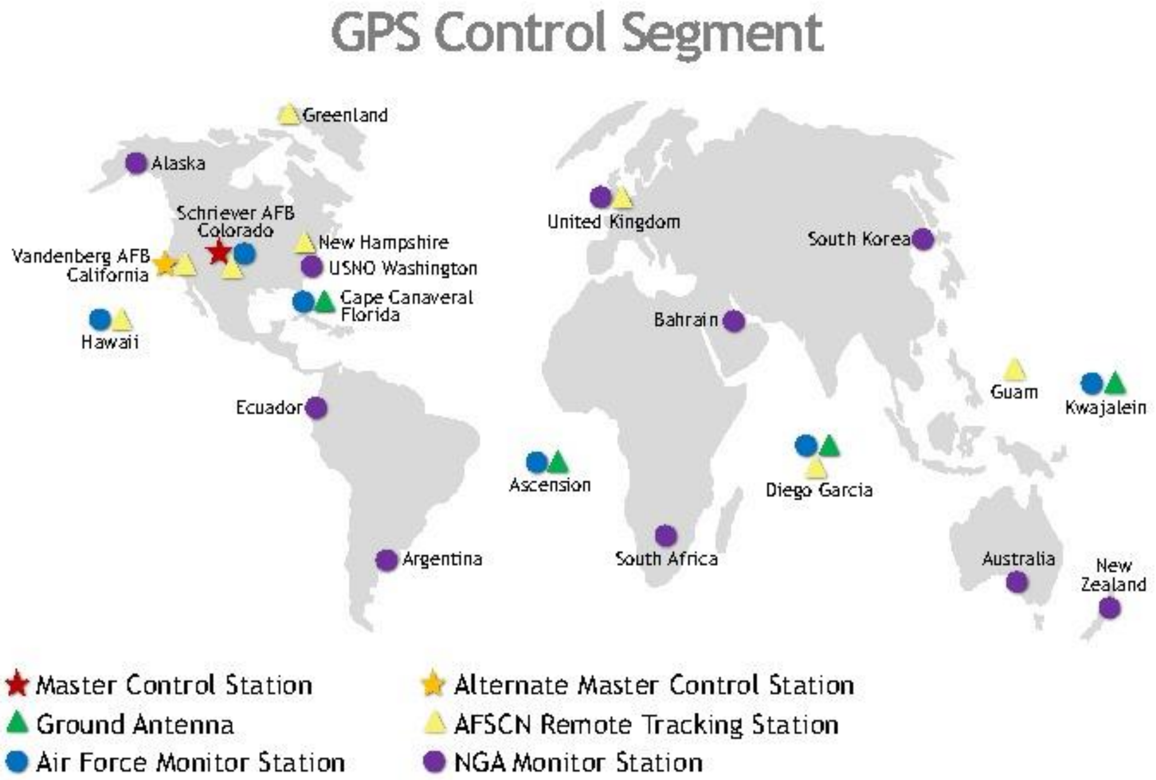




- 24 Satellites (now>30)
- 6 orbital planes, 55^{o} inclination
- Period: half a sidereal day
- Several atomic clocks/satellite
- Several spare satellites

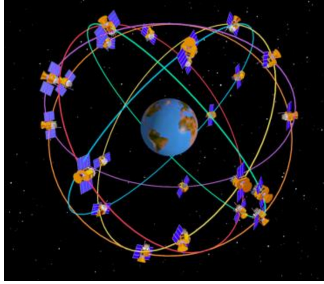
Control Segment

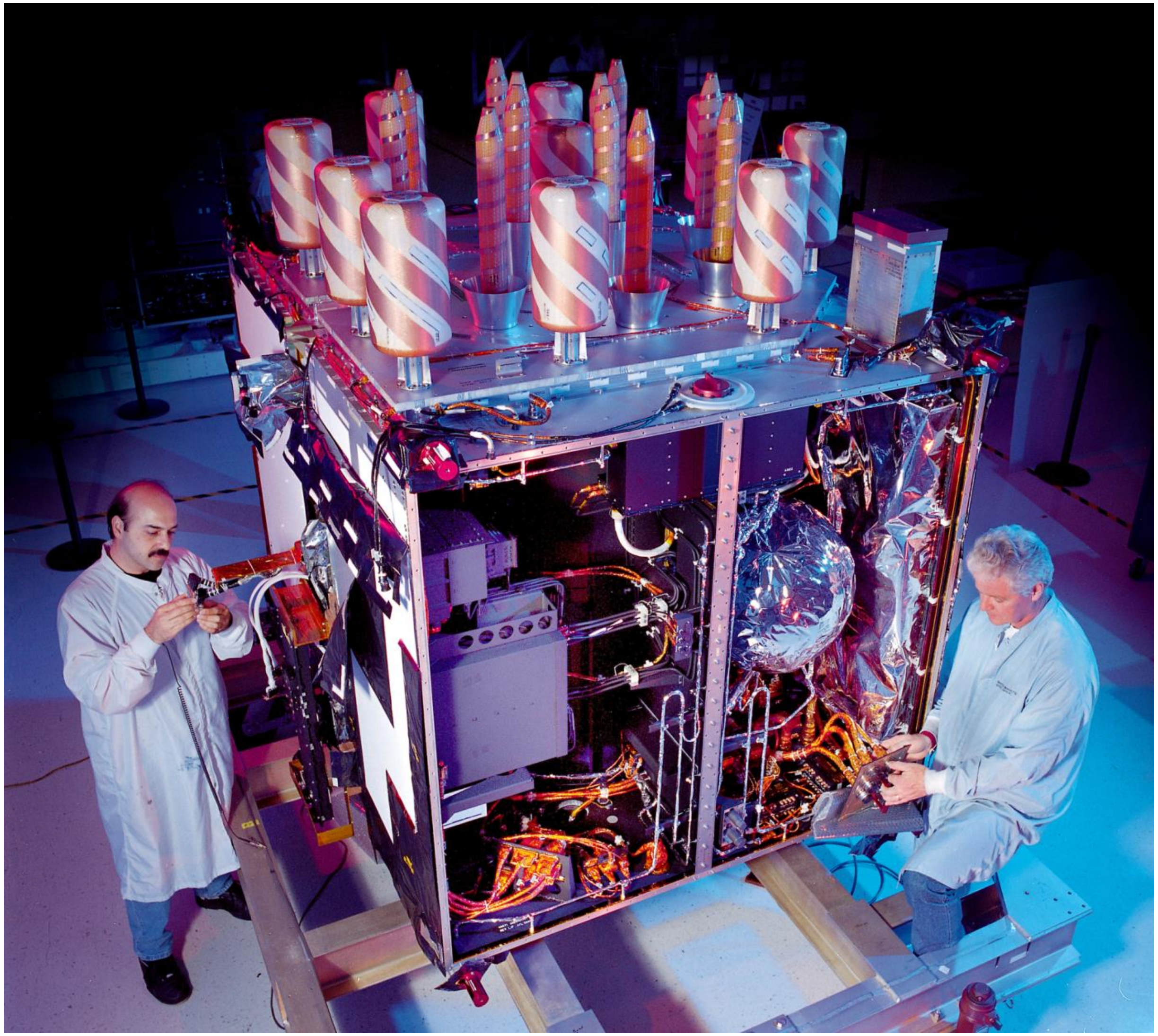




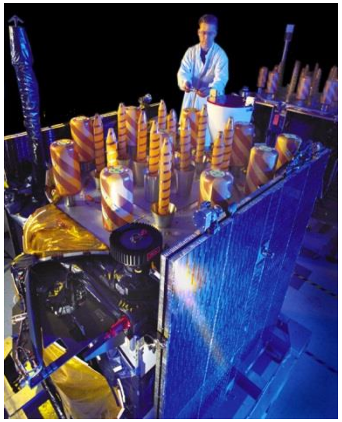
Updated April 2014

GPS IIR Satellite



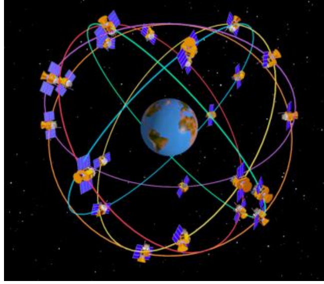


IIF

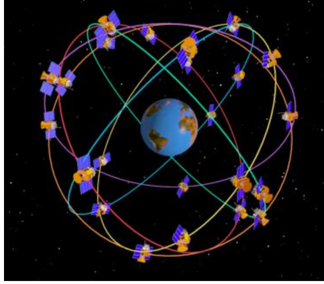


III

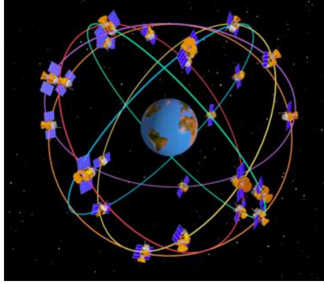
Block III satellite



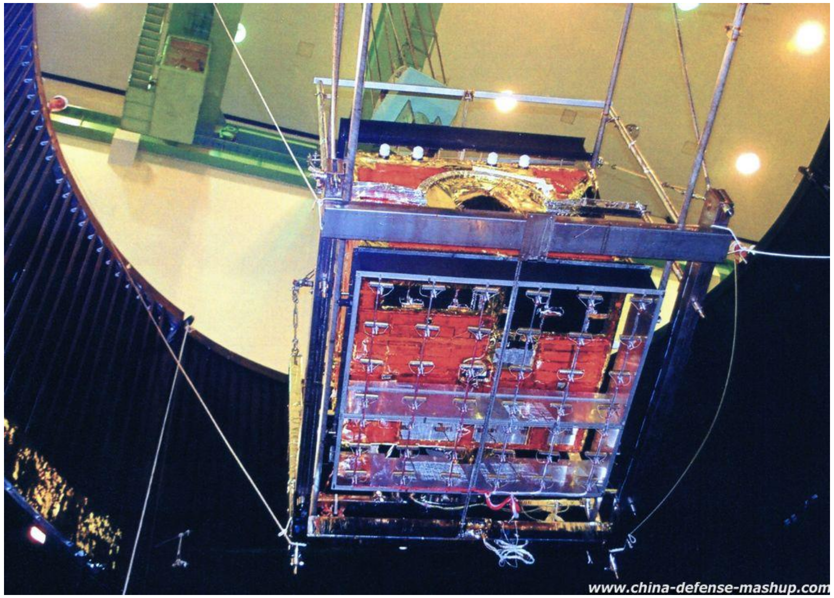
Block III GPS satellite



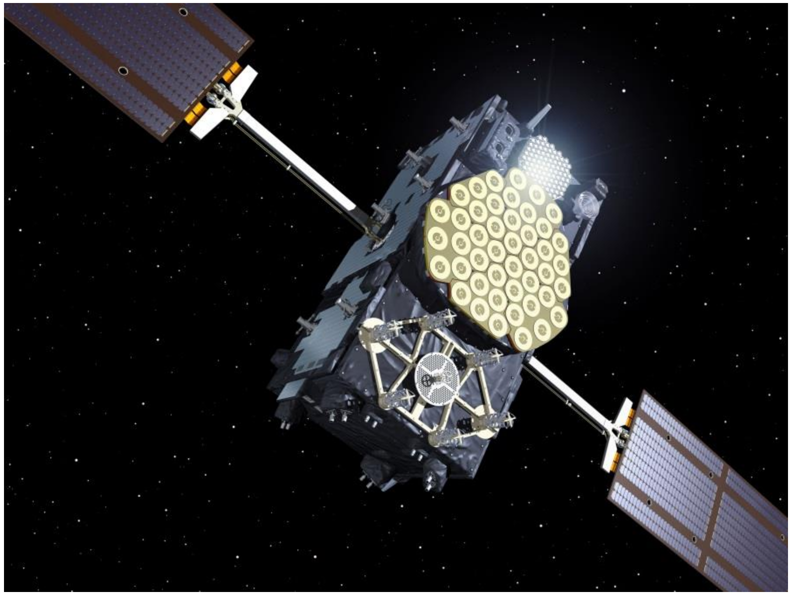
Other GNSS Satellites



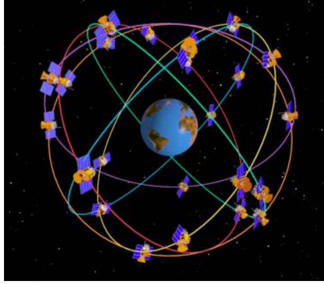
Beidou



GALILEO



Constellation Status-GPS



Real-Time GPS monitoring

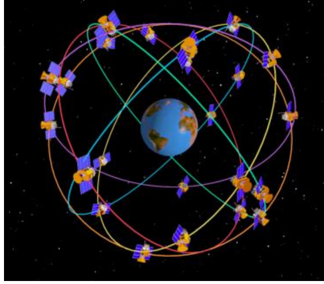
Evaluation of characteristics

Total satellites in constellation	32 SC
Operational	31 SC
In commissioning phase	-
In maintenance	1 SC
In decommissioning phase	-

GPS constellation status for 20.10.15 under the analysis of the almanac accepted in IAC

Plane	Slot	PRN	NORAD	Type SC	Launch date	Input date	Outage date	Life-time (months)	Notes
A									
	2	31	29486	IIR-M	25.09.06	13.10.06		108.3	
	4	7	32711	IIR-M	15.03.08	24.03.08		90.9	
	5	24	38833	II-F	04.10.12	14.11.12		35.2	
	6	30	39533	II-F	21.02.14	30.05.14		16.7	
B	1	16	27663	II-R	29.01.03	18.02.03		152.1	
	2	25	36585	II-F	28.05.10	27.08.10		61.8	
	3	28	26407	II-R	16.07.00	17.08.00		182.2	
	4	12	29601	IIR-M	17.11.06	13.12.06		106.3	
	5	26	40534	II-F	25.03.15	20.04.15		6.0	
	6		34661	IIR-M	24.03.09				Flight Tests
C	1	29	32384	IIR-M	20.12.07	02.01.08		93.6	
	2	27	39166	II-F	15.05.13	21.06.13		28.0	
	3	19	28190	II-R	20.03.04	05.04.04		138.6	
	4	17	28874	IIR-M	26.09.05	13.11.05		119.3	
	5	8	40730	II-F	15.07.15	12.08.15		2.3	
D	1	2	28474	II-R	06.11.04	22.11.04		131.0	
	2	1	37753	II-F	16.07.11	14.10.11		48.2	
	3	21	27704	II-R	31.03.03	12.04.03		150.4	
	4	4	22877	II-A	26.10.93	22.11.93		263.1	
	5	11	25933	II-R	07.10.99	03.01.00		189.7	
	6	6	39741	II-F	17.05.14	10.06.14		16.3	
E	1	20	26360	II-R	11.05.00	01.06.00		184.7	
	2	22	28129	II-R	21.12.03	12.01.04		141.3	
	3	5	35752	IIR-M	17.08.09	27.08.09		73.8	
	4	18	26690	II-R	30.01.01	15.02.01		176.2	
	5	32	20959	II-A	26.11.90	10.12.90		298.5	
	1	3	40294	II-F	29.10.14	12.12.14		10.3	
	1	14	26605	II-R	10.11.00	10.12.00		178.4	
	2	15	32260	IIR-M	17.10.07	31.10.07		95.7	
	3	13	24876	II-R	23.07.97	31.01.98		212.7	

GPS Constellation Status



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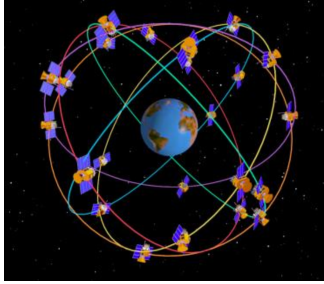
SUBJ: GPS STATUS      11 JUN 2017

1. SATELLITES, PLANES, AND CLOCKS (CS=CESIUM RB=RUBIDIUM):
A. BLOCK I : NONE
B. BLOCK II: PRNS  1,  2,  3,  5,  6,  7,  8,  9, 10, 11, 12, 13, 14, 15
   PLANE   : SLOT D2, D1, E1, E3, D4, A4, C3, F3, E2, D5, B4, F6, F1, F2
   CLOCK   :      RB, RB, RB, RB, RB, RB, CS, RB, RB, RB, RB, RB, RB, RB
   BLOCK II: PRNS 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29
   PLANE   : SLOT B1, C4, E4, C5, B6, D3, E6, F4, A1, B2, B5, C2, B3, C1
   CLOCK   :      RB, RB, RB, RB, RB, RB, RB, RB, CS, RB, RB, RB, RB, RB
   BLOCK II: PRNS 30, 31, 32
   PLANE   : SLOT A3, A2, F5
   CLOCK   :      RB, RB, RB
```

GALILEO Constellation Status 11/21/2016



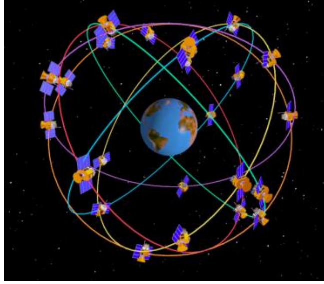
Reference Constellation Orbital and Technical Parameters [1]

Satellite	SV ID	Slot	Launch Date[4]	Semi-Major Axis (Km)	Eccentricity	Inclination (deg)	RAAN (deg)[2]	Arg. Perigee (deg)[2]	Mean Anomaly (deg)[2,3]
Nominal Slots									
GSAT0101	11	B05	2011-10-21	29599.8	0.0	56.0	77.632	0.0	15.153
GSAT0102	12	B06	2011-10-21	29599.8	0.0	56.0	77.632	0.0	60.153
GSAT0103	19	C04	2012-10-12	29599.8	0.0	56.0	197.632	0.0	345.153
GSAT0104	20	C05	2012-	29599.8	0.0	56.0	197.632	0.0	30.153
			05-24						
GSAT0211	02	A06	2016-05-24	29599.8	0.0	56.0	317.632	0.0	45.153
GSAT0207	07	C06	2016-11-17	29599.8	0.0	56.0	197.632	0.0	75.153
GSAT0212	03	C08	2016-11-17	29599.8	0.0	56.0	197.632	0.0	165.153
GSAT0213	04	C03	2016-11-17	29599.8	0.0	56.0	197.6316	0.0	300.153
GSAT0214	05	C01	2016-11-17	29599.8	0.0	56.0	197.632	0.0	210.153
Extended Slots									
GSAT0201	18	Ext01	2014-08-22	27977.6	0.162	49.850	52.521	56.198	316.069
GSAT0202	14	Ext02	2014-08-22	27977.6	0.162	49.850	52.521	56.198	136.069

Extended Slots

GSAT0201	18	Ext01	2014-08-22	27977.6	0.162	49.850	52.521	56.198	316.069
GSAT0202	14	Ext02	2014-08-22	27977.6	0.162	49.850	52.521	56.198	136.069

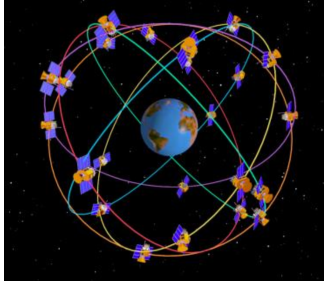
Planned Beidou Constellation



Orbit parmts.	GEO	IGSO	MEO
Semi-Major Axis (Km)	42164	42164	27878
Eccentricity	0	0	0
Inclination (deg)	0	55	55
RAAN (deg)	158.75E, 180E, 210.5E, 240E,260E	218E,98E,338E	
Argument Perigee	0	0	--
Mean anomaly (deg)	0	218E:0,98E:120,338E:240	
# Sats	5	3	27
# Planes	1	3	3

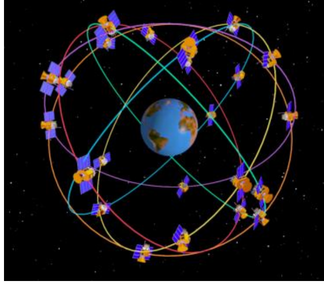
Final BeiDou Constellation

Beidou Constellation Status



- As of November 2016: 20 operational satellites of 35 planned
  - 6 satellites in geostationary orbits;
  - 8 in 55-degree inclined geosynchronous orbits;
  - 6 in medium earth orbits at altitude 21,500 km

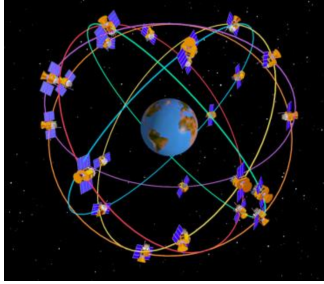
Fundamental Principles

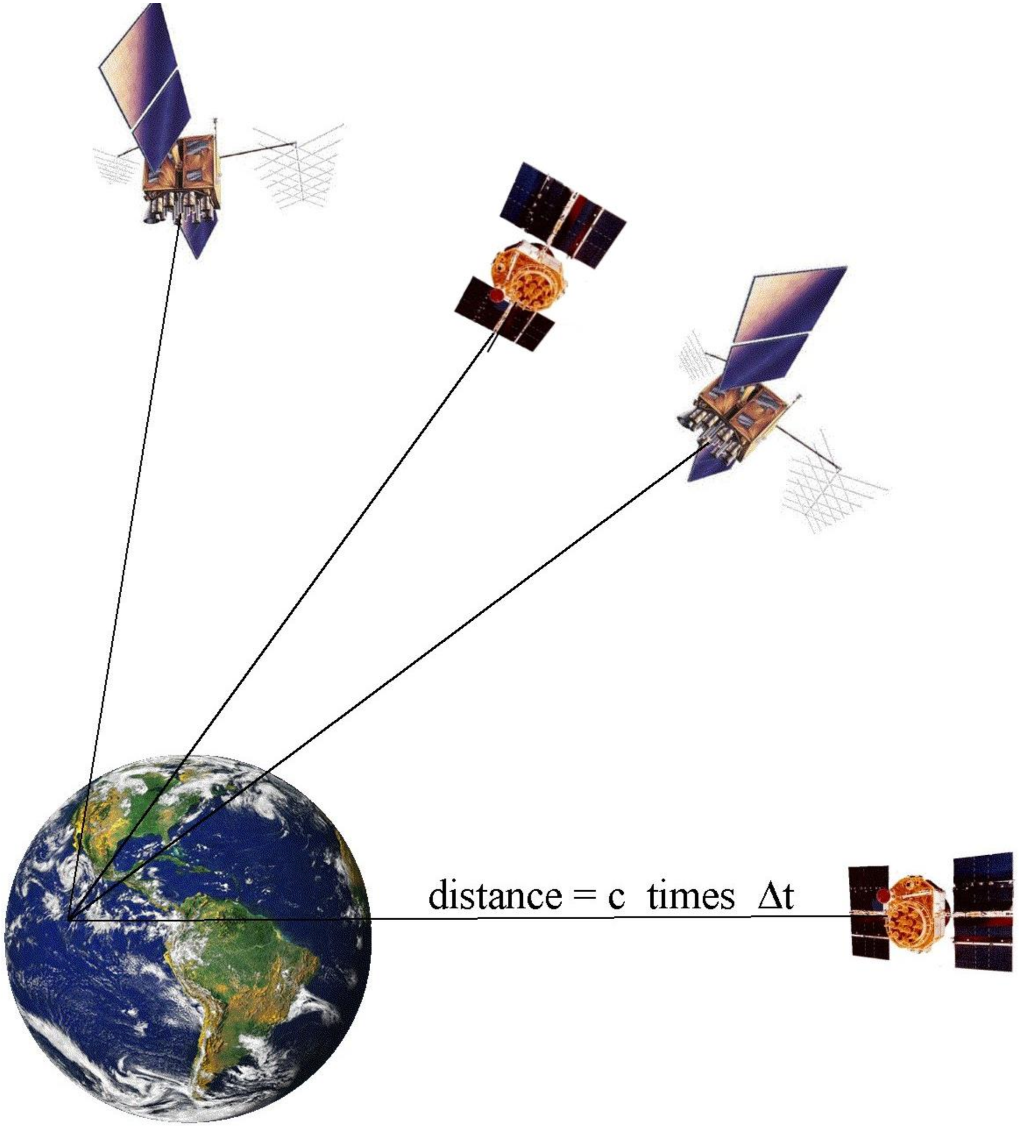


- Constancy of the speed of light
  - The speed of light, c, is a constant independent of the motion of the source (or of the observer);

- Principle of Equivalence (“weak form”)
  - Over a small region of space and time, the fictitious gravitational field induced by acceleration cannot be distinguished from a real gravitational field due to a mass.

Constancy of c



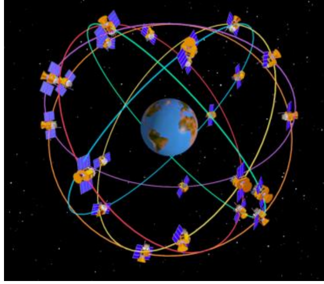


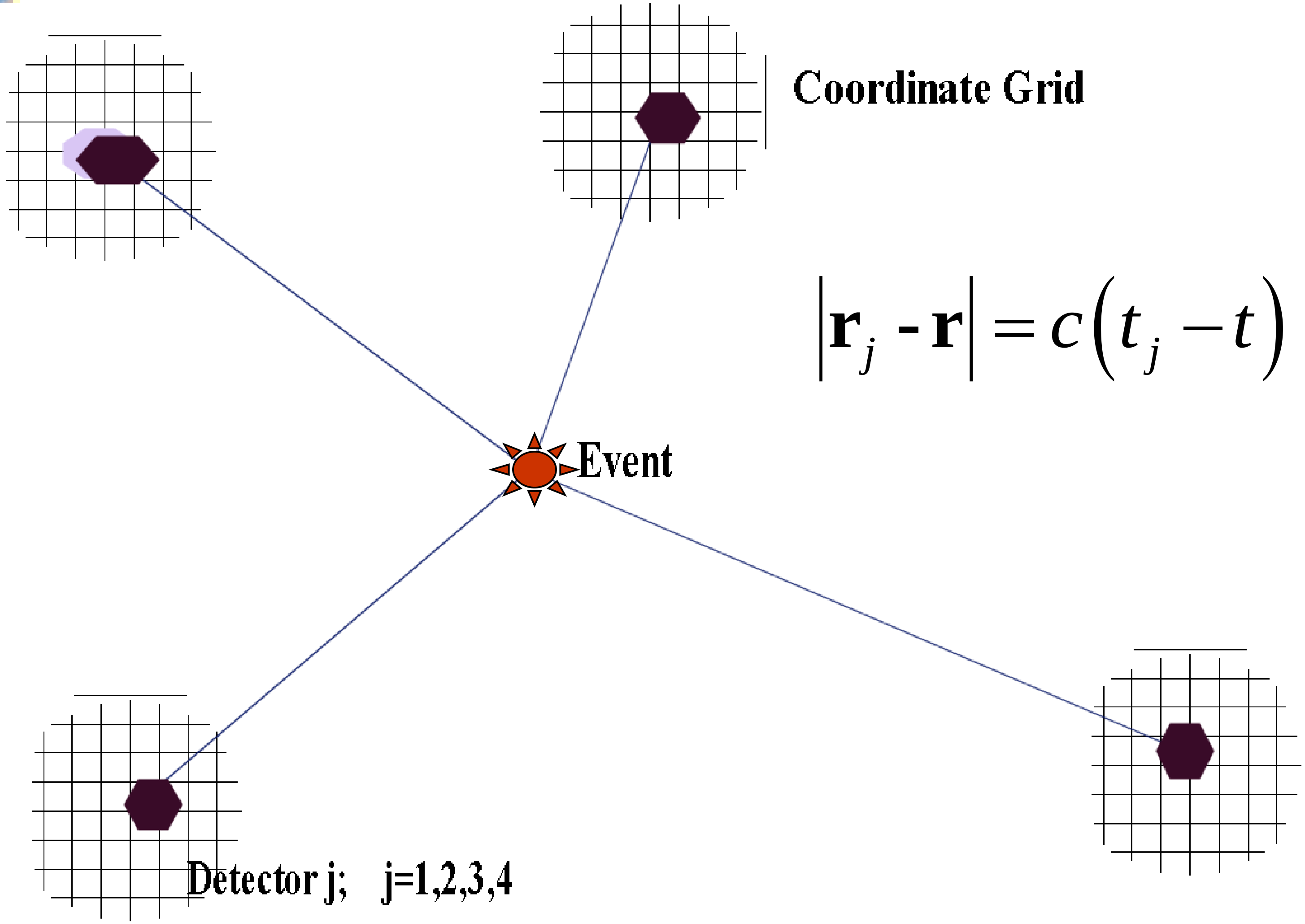
$$\left|\mathbf{r}-\mathbf{r}_j\right| = c\left(t - t_j\right),$$

$$j = 1, 2, 3, 4$$

Synchronization
Is the key!

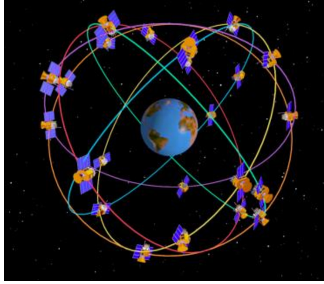
Reciprocity

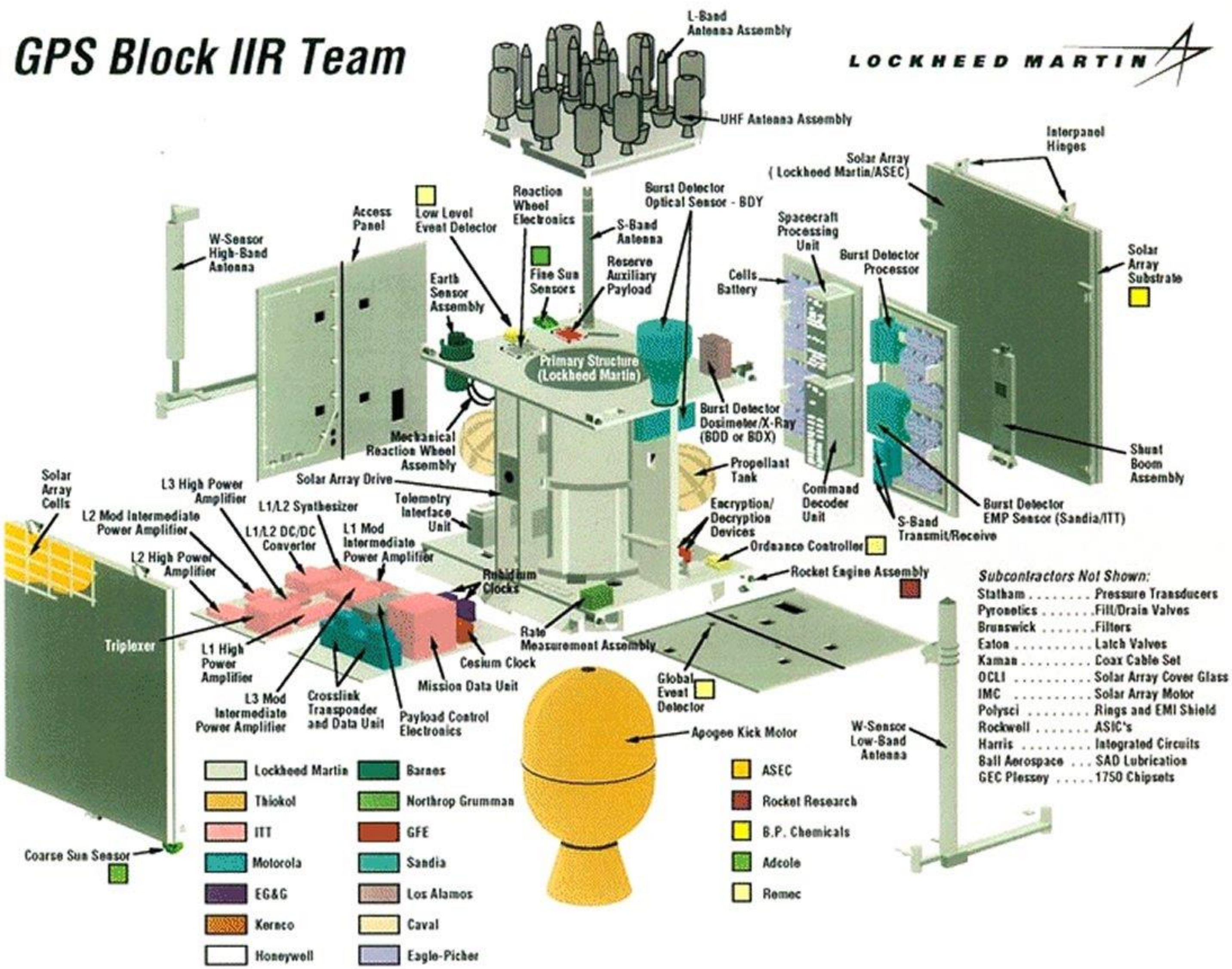




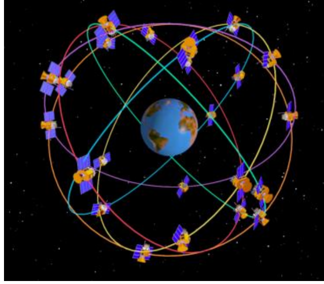
$$|\mathbf{r}_j - \mathbf{r}| = c\left(t_j - t\right)$$

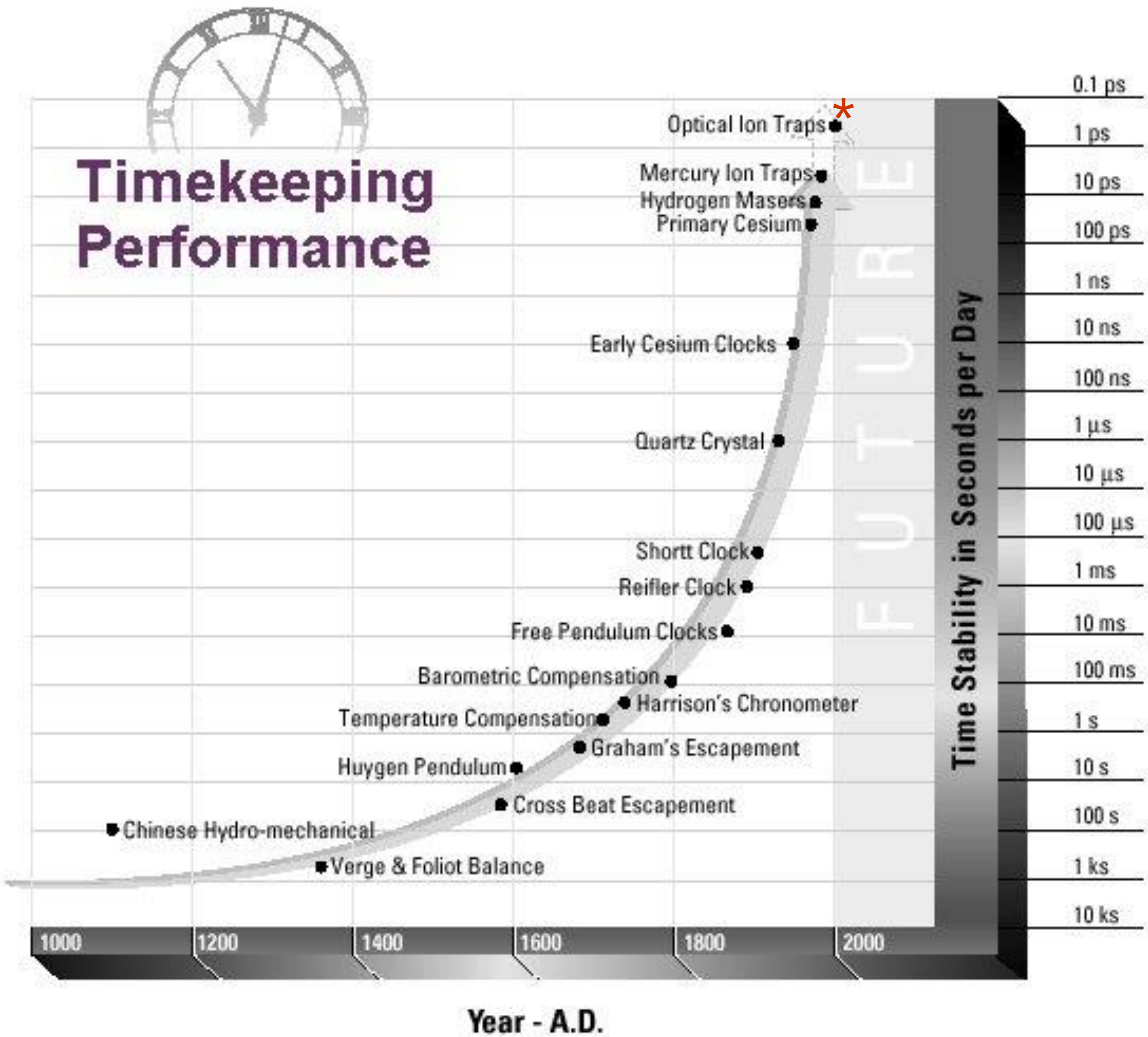
Exploded Block IIR Satellite View





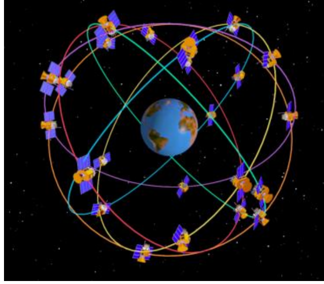
Clock Improvement Since 1000 A.D.





1 ns/day= 10^{-14}

Why are atomic clocks needed?



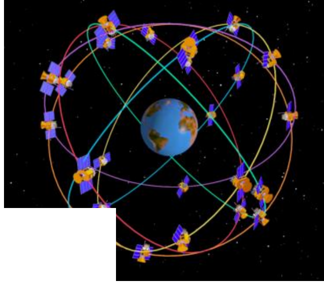
To reduce the effect of clock error to < 2 meters,

the clock error must be less than 2/c = 6.7×10^{-9} sec.

Half a day = 43200 seconds, so the fractional clock error must be less than:

(2 m)/(43200 s x c) = $\mathbf{1.5 \times 10^{-13}}$.

Only atomic clocks can achieve such stability.



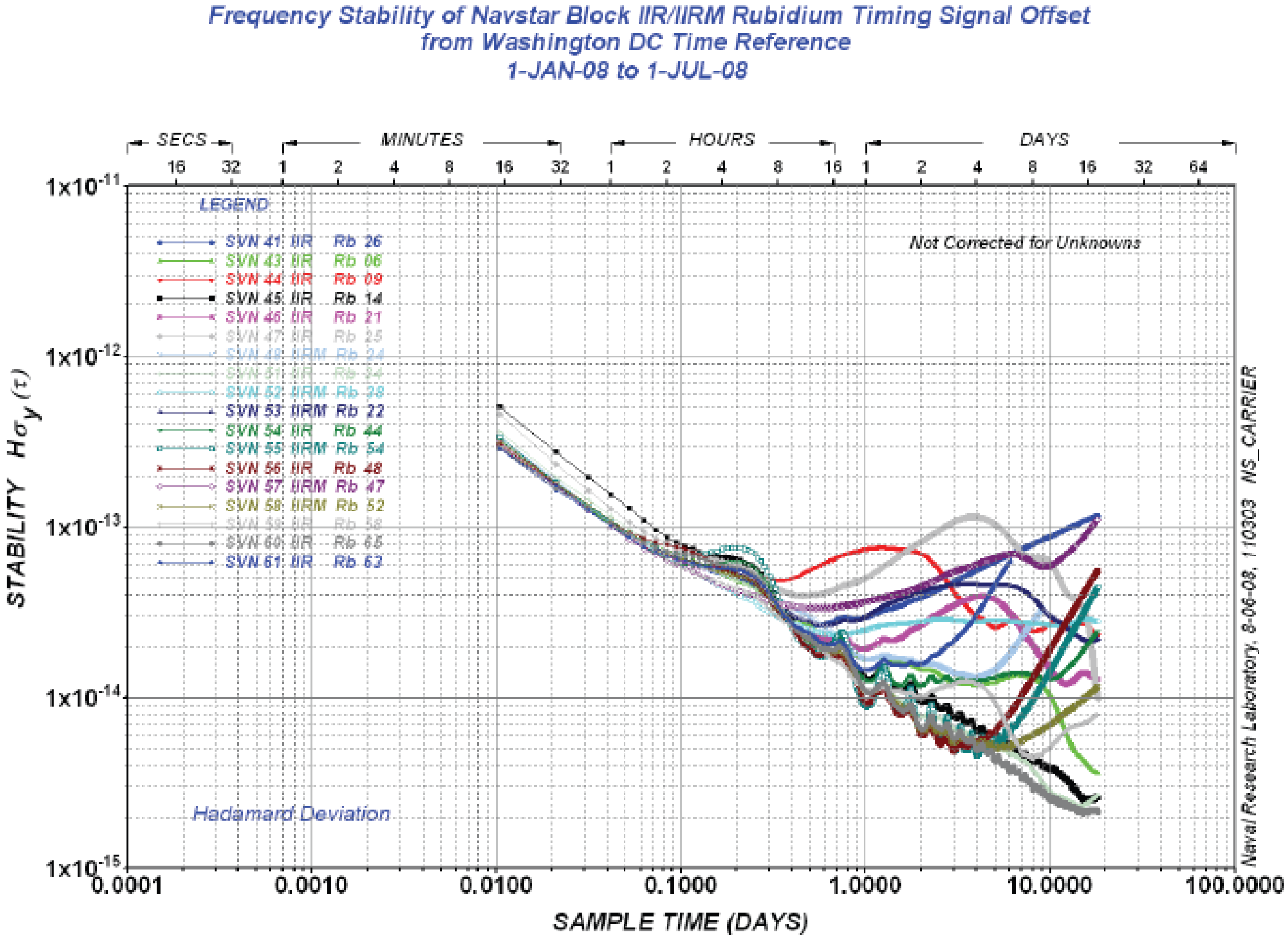
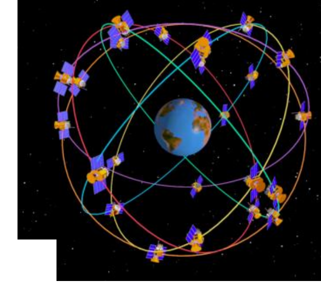
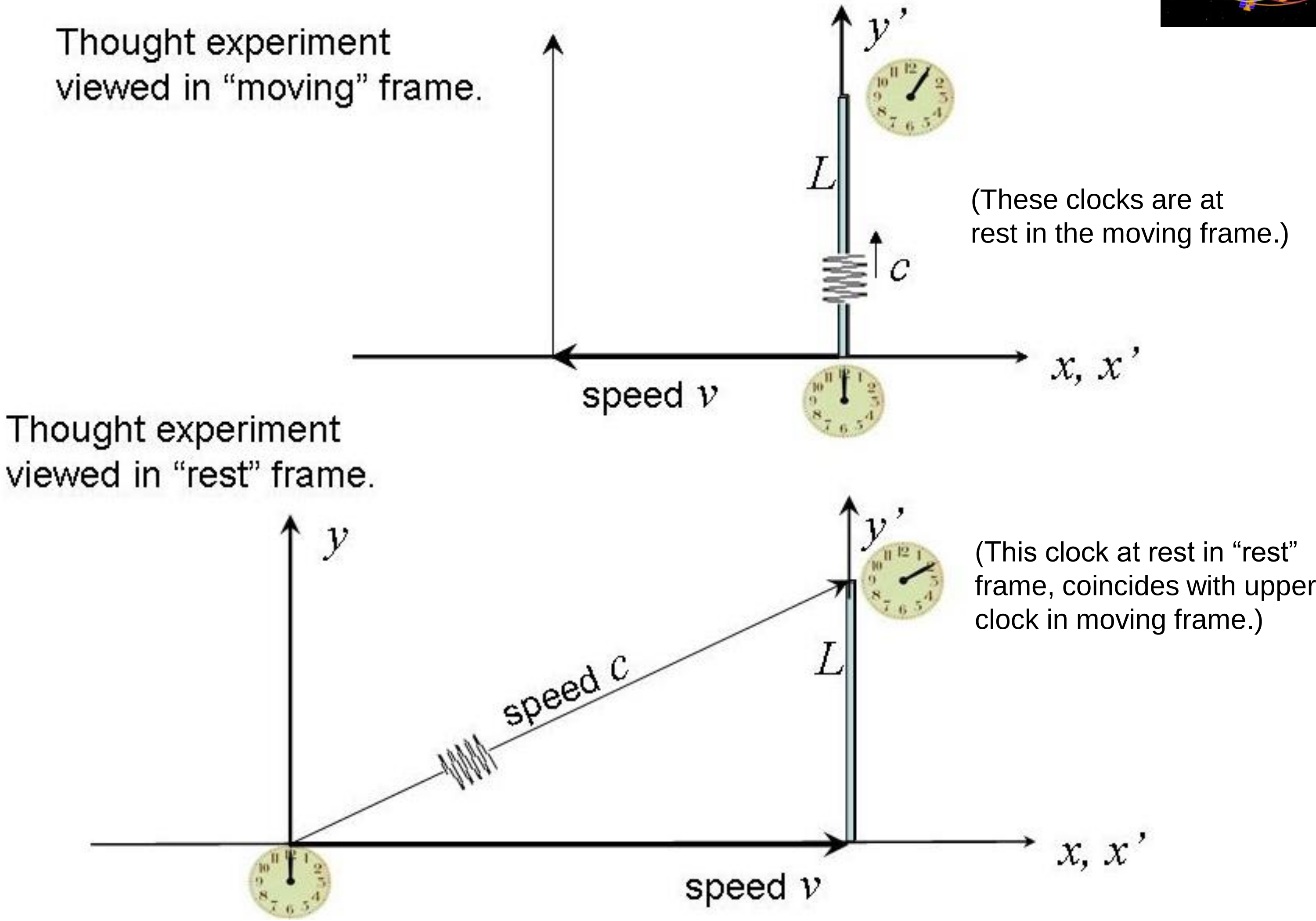


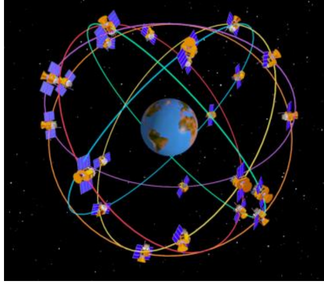
Figure 2-7. Frequency Stability Profiles (Hadamard) of Block IIR/IIR-M Rubidium Clocks

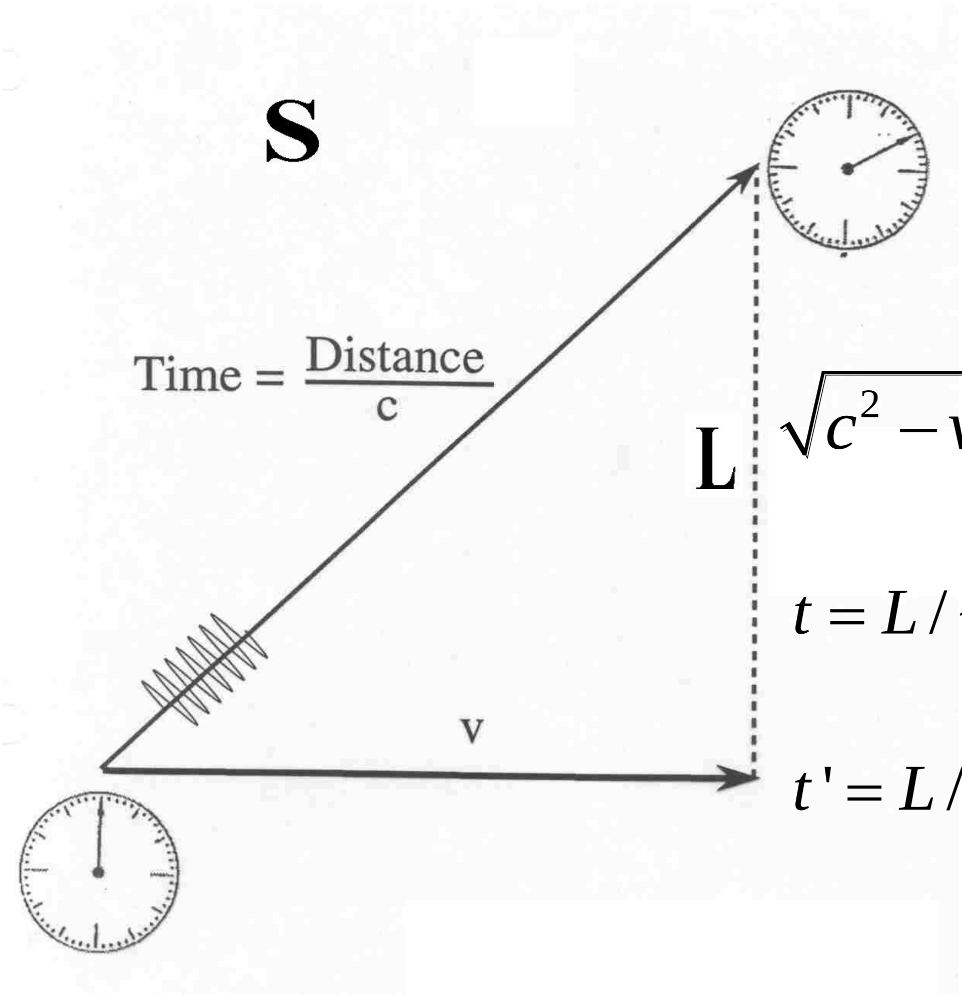
Constancy of the speed of light implies time dilation





Einstein's Light Clock



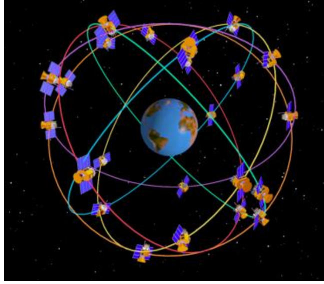


$$\sqrt{c^2 - v^2}$$

$$t = L / \sqrt{c^2 - v^2} = \frac{L/c}{\sqrt{1 - v^2/c^2}};$$

$$t' = L/c = \sqrt{1 - v^2/c^2}\, t$$

How Big is Time Dilation?



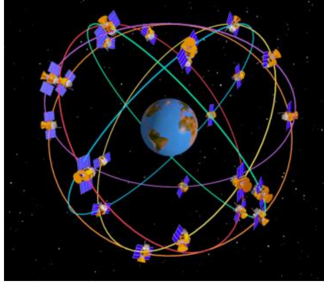
$$\sqrt{1-v^2/c^2} \approx 1-\frac{1}{2}\frac{v^2}{c^2};$$

$$v = 4000\,\text{m/s};$$

$$-\frac{1}{2}\frac{v^2}{c^2} = -8.9\times 10^{-11}$$

(about 8 microseconds per day)

Accounting For Relativistic Effects



Example: Time Dilation:

$$d\tau = \sqrt{1 - v^2 / c^2}\, dt;$$

$$dt = \left(1 - v^2 / c^2\right)^{-1/2} d\tau$$

$$\cong \left(1 + \frac{1}{2}\frac{v^2}{c^2}\right) d\tau.$$

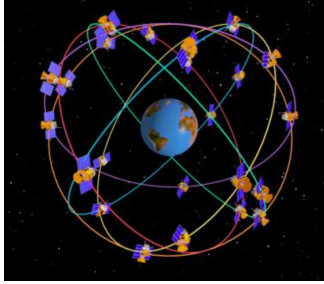
Elapsed Coordinate time:

$$\Delta t = \int_{\text{path}} d\tau \left(1 + \frac{1}{2}\frac{v^2}{c^2}\right)$$

Observed Proper Time ($d\tau$)

Note: $(cd\tau)^2 = (cdt)^2 (1 - v^2 / c^2) = (cdt)^2 - dx^2 - dy^2 - dz^2$

Fundamental line element



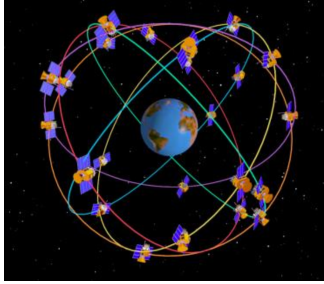
For light: $$0 = ds^2 = (cdt)^2 - dx^2 - dy^2 - dz^2$$

Time dilation: $$ds^2 = (cd\tau)^2 = (cdt)^2\left(1 - \frac{1}{c^2}\frac{dx^2 + dy^2 + dz^2}{dt^2}\right)$$
$$= (cdt)^2 - dx^2 - dy^2 - dz^2$$

With gravity: $$ds^2 = \left(1 + \frac{2\Phi}{c^2}\right)(cdt)^2 - \left(1 - \frac{2\Phi}{c^2}\right)\left(dx^2 + dy^2 + dz^2\right)$$

Motion of Planets: $$\delta\int_{path} ds = 0.$$

Coordinate Time

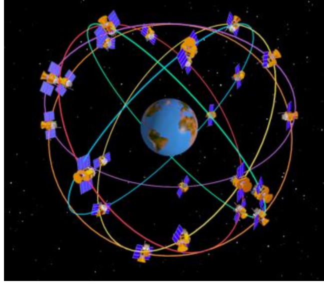


- In special relativity:
  To each real clock, corrections are applied such that at each instant, the clock would read the same as a hypothetical clock at rest at the same point in the underlying inertial frame.

- When gravitational fields are present:
  Additional corrections compensate for gravitational frequency shifts relative to a reference on earth's geoid.

- GPS time is an example of coordinate time, in which the reference is on the earth's rotating geoid.

Sagnac Effect



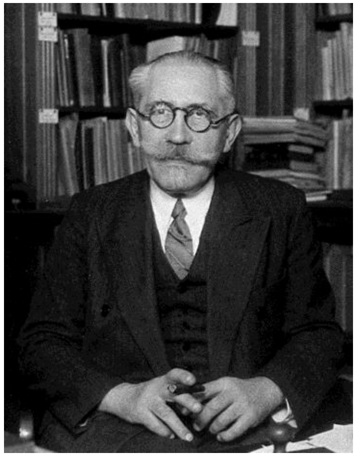
$$(cd\tau)^2 = ds^2 = (cdt)^2 - dr^2 - r^2(d\phi_{ECI})^2 - dz^2$$

In a rotating coordinate system such as one fixed to the earth, let the axis representing the zero for the angle ϕ rotate with constant angular speed:

$$ds^2 = (c\,dt)^2 - dr^2 - r^2(d(\phi - \omega t))^2 - dz^2$$

$$= \left(1 - \frac{\omega^2 r^2}{c^2}\right)(c\,dt)^2 + 2\omega r^2 d\phi dt - dr^2 - r^2 d\phi^2 - dz^2$$

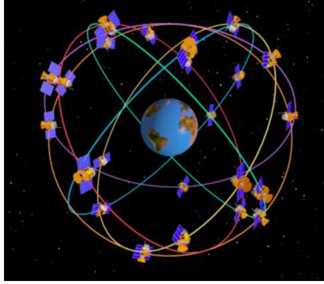
This is the Langevin metric.

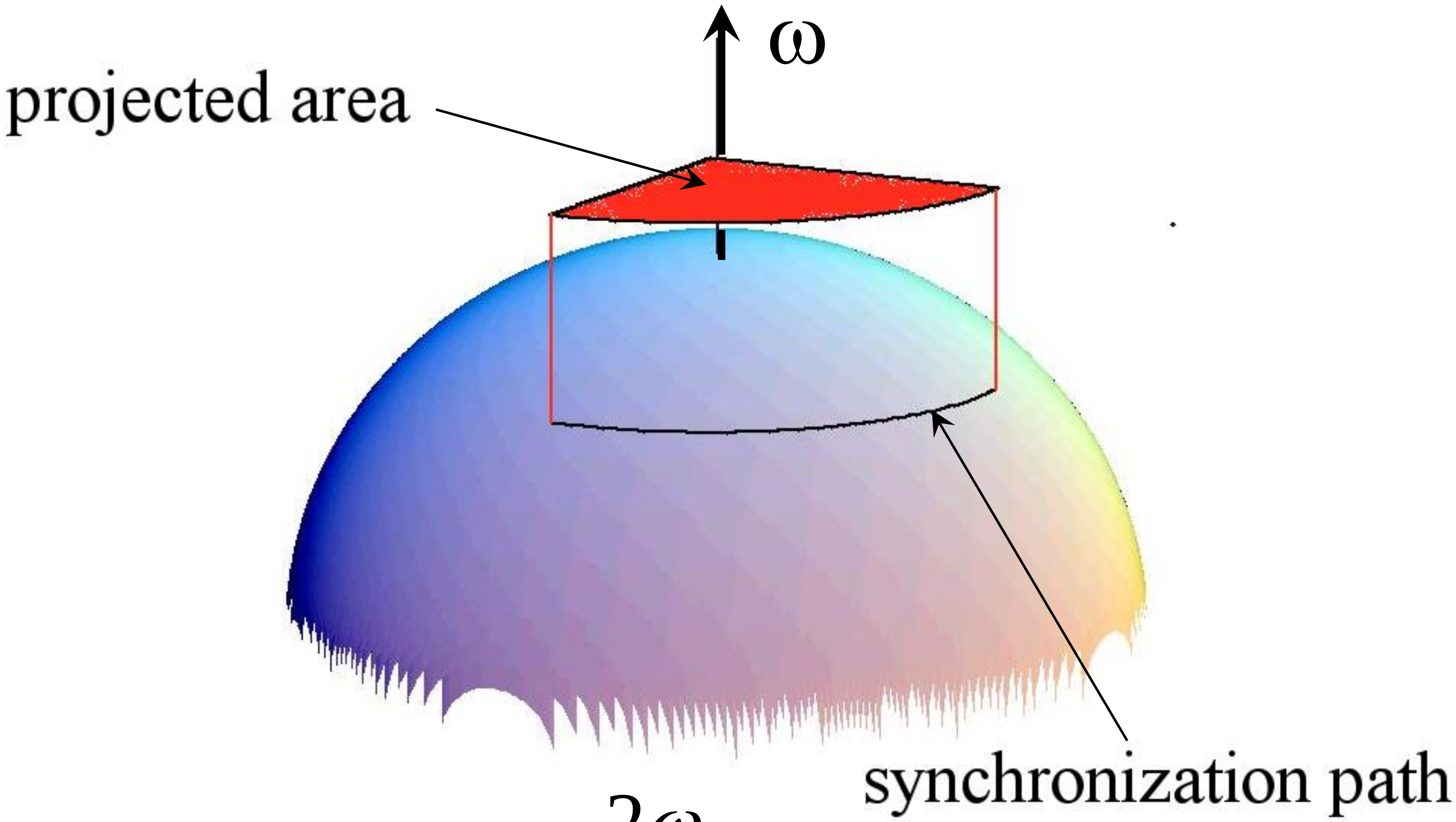


For light: solving for dt to first order in ω, the $d\phi dt$ term gives rise to the Sagnac effect.

$$dt = \frac{d\sigma}{c} + \frac{\omega r^2 d\phi}{c^2}$$

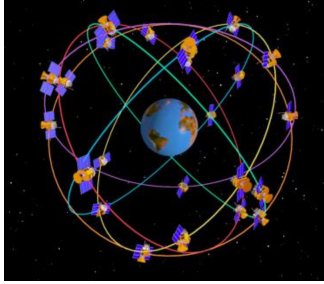
Sagnac Effect on Synchronization in a Rotating System

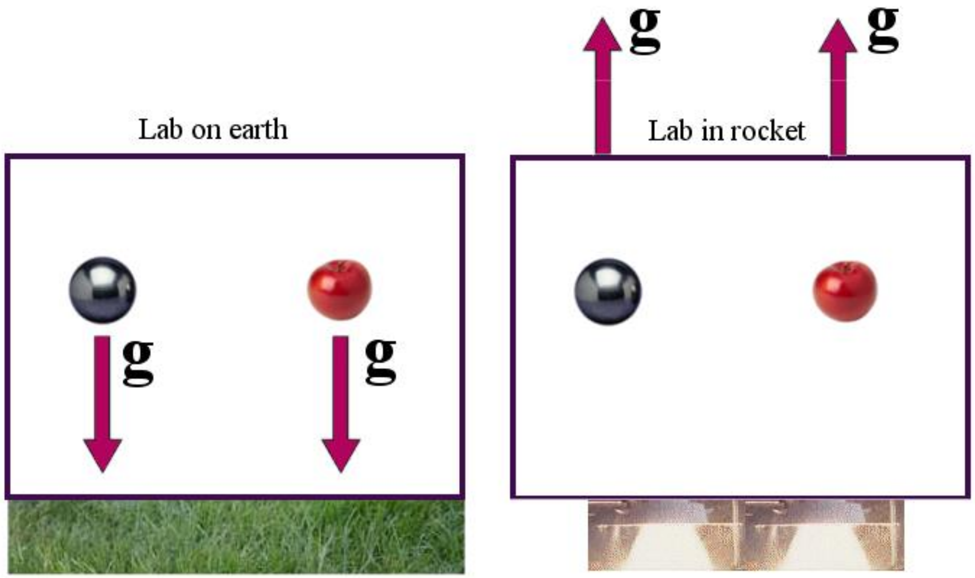




Sagnac Correction = $\dfrac{2\omega}{c^2} A_z$

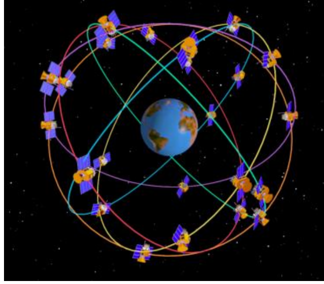
Equivalence principle and gravitational frequency shifts

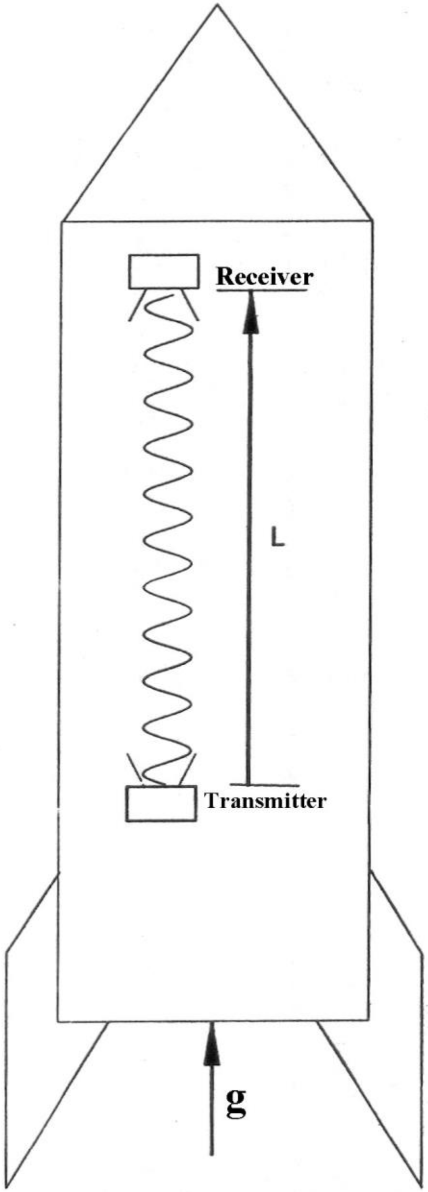




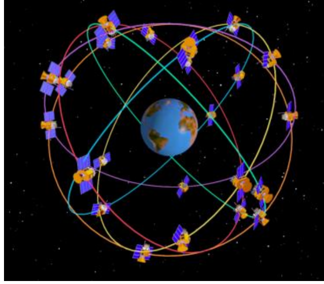
Over a small region of space and time, a fictitious gravity field induced by acceleration cannot be distinguished From a gravity field produced by mass.

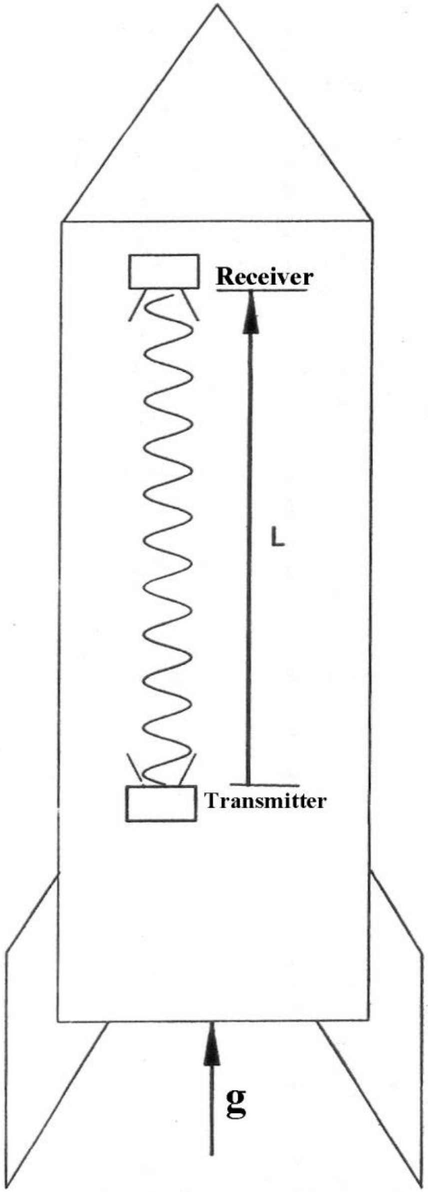
Gravitational Frequency Shift





Gravitational Frequency Shift



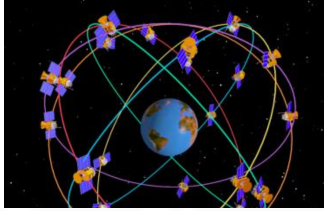


$$t = L / c;$$

$$v = gt = \frac{gL}{c};$$

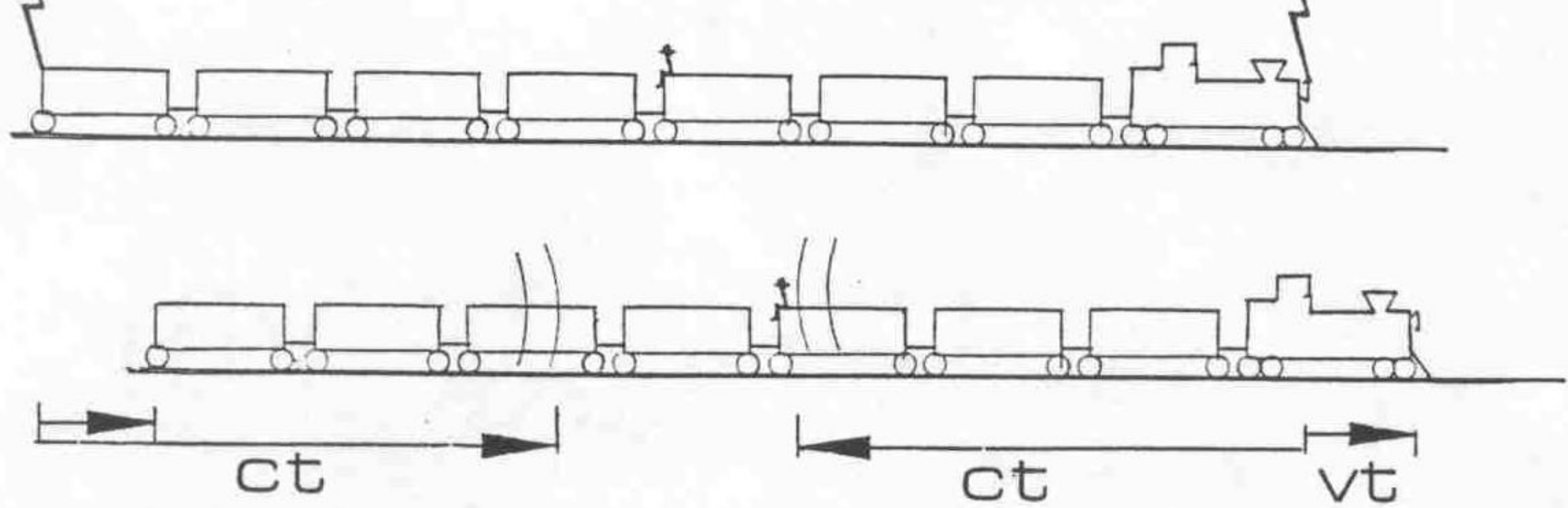
$$\frac{\Delta f}{f} = -\frac{v}{c} = -\frac{gL}{c^2} = -\frac{\Delta\Phi}{c^2}.$$

Relativity of Simultaneity



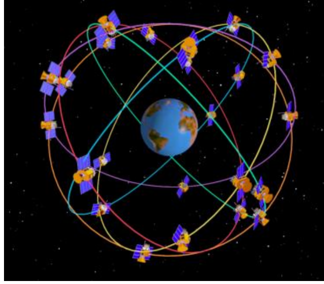
To an observer on the ground, let two lightning strokes at the front and back of the train be simultaneous.

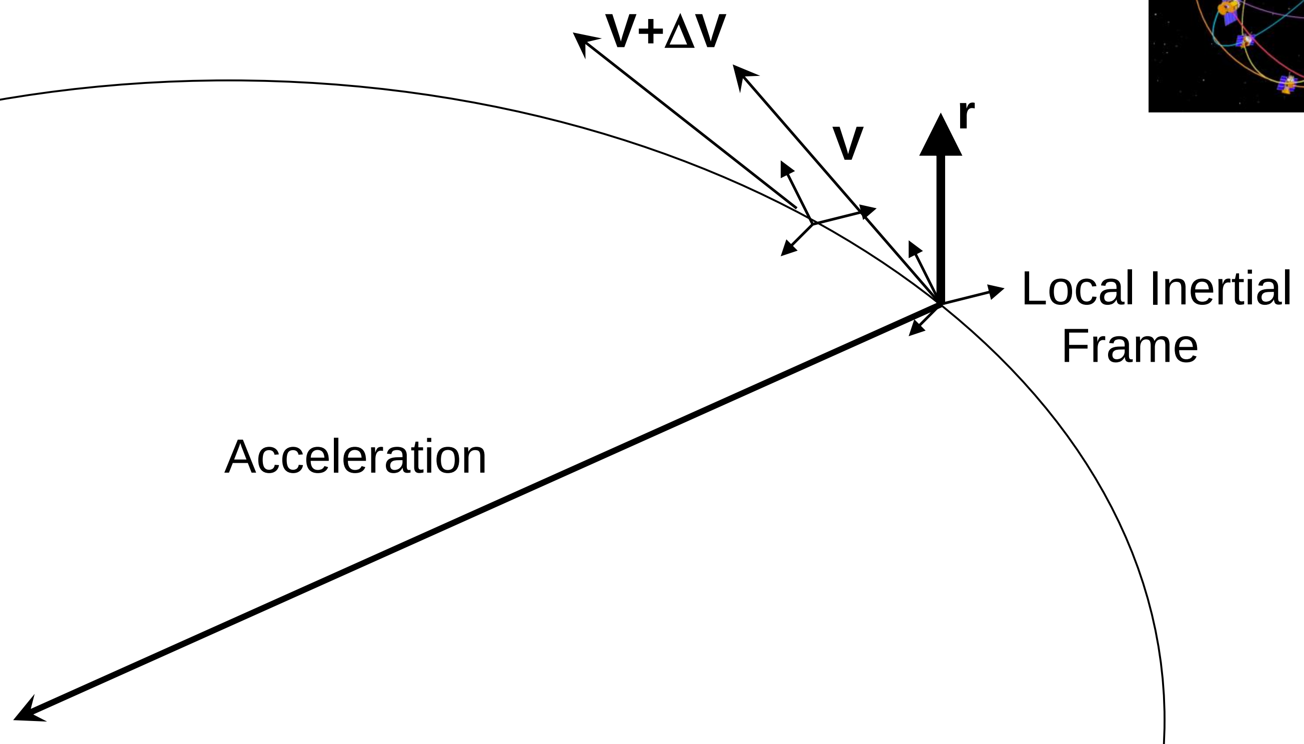
The “moving” observer at the train’s midpoint finds the event at front occurs first.

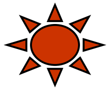


$$t' \approx t - \frac{vx}{c^2} \Rightarrow t - \frac{\mathbf{V}\cdot\mathbf{r}}{c^2}$$

Principle of Equivalence





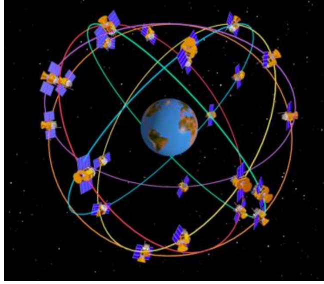


$$\text{Induced potential difference}/c^2 = \frac{\mathbf{A}\bullet\mathbf{r}}{c^2} = \frac{\Delta\mathbf{V}\bullet\mathbf{r}}{\Delta T c^2};$$

$$\text{Gravitational potential difference}/c^2 = \frac{\nabla\Phi\bullet\mathbf{r}}{c^2};$$

$$\text{Net potential difference}/c^2 = \frac{\mathbf{A}\bullet\mathbf{r}+\nabla\Phi\bullet\mathbf{r}}{c^2} = 0;$$

Fundamental Scalar Invariant



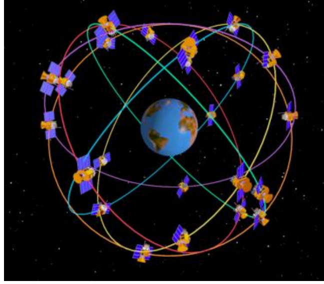
$$(c\,d\tau)^2 = \left(1+\frac{2\Phi}{c^2}\right)(c\,dt)^2 - \left(1-\frac{2\Phi}{c^2}\right)(dx^2+dy^2+dz^2)$$

$$\Phi = -\frac{GM}{r}\left(1-\frac{J_2 a_1^2}{r^2}\left[\frac{3z^2}{r^2}-\frac{1}{2}\right]\right)$$

For a clock near earth,

$$\Delta t = \int_{\text{path}} d\tau\left[1+\frac{1}{2}\frac{v^2}{c^2}-\frac{\Phi}{c^2}\right]$$

Earth-fixed Clock



$$\Delta t = \int_{\text{path}} d\tau \left[1 - \frac{\Phi_0}{c^2} \right];$$

$$\frac{\Phi_0}{c^2} = -\frac{GM}{c^2 a_1} - \frac{GMJ_2}{2c^2 a_1} - \frac{\omega^2 a_1^2}{2c^2}$$

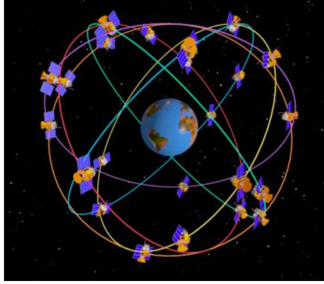
$$= \left(-6.95348 - .00376 - .01203 \right) \times 10^{-10}$$

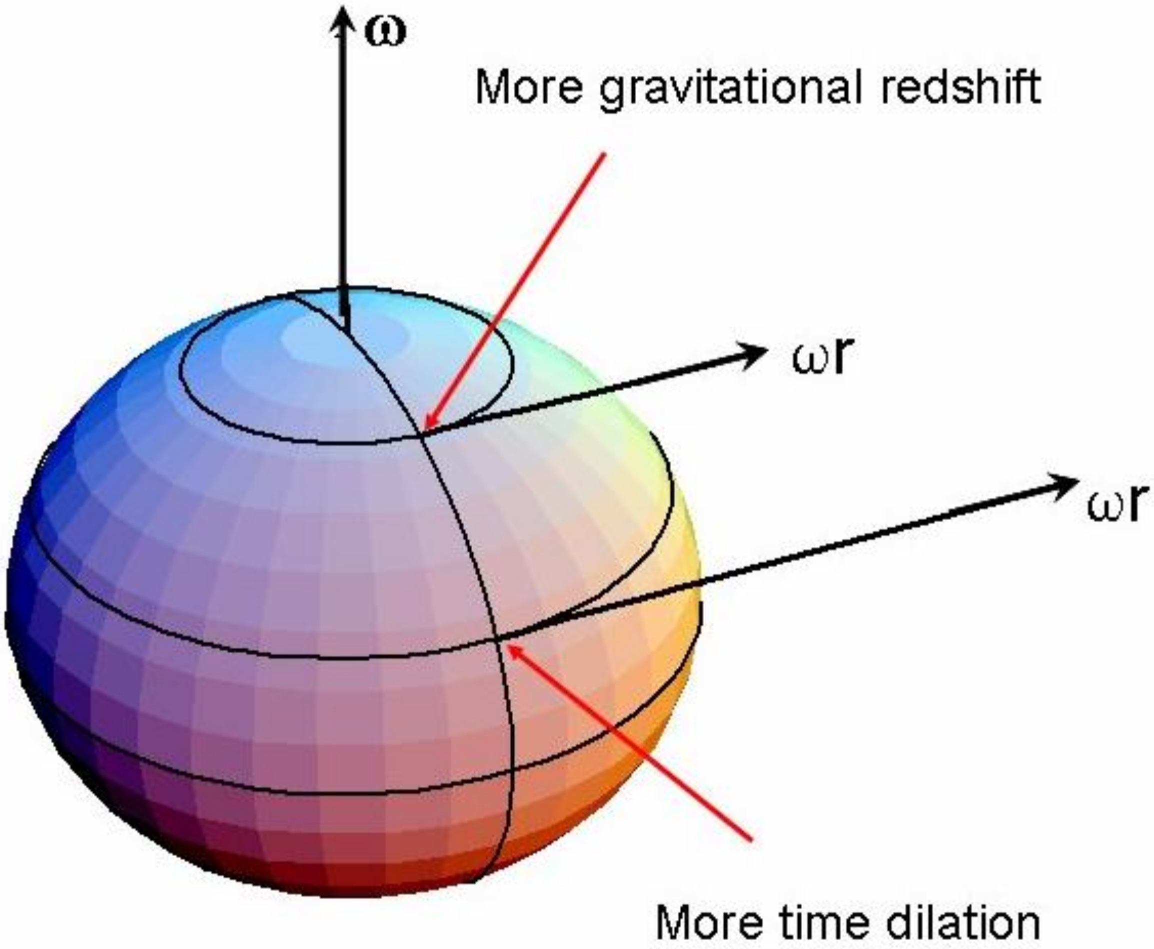
$$= -6.96927 \times 10^{-10}$$

Note about centripetal term

This is the fractional frequency shift of an atomic clock fixed on earth, relative to an atomic clock at infinity.

Clocks on earth's geoid beat at equal rates



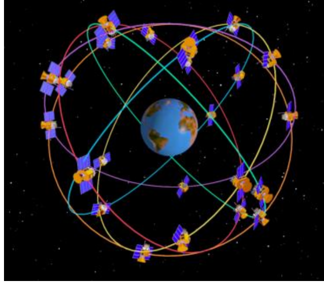


Clocks at rest on geoid beat at equal rates, defining International Atomic Time.

They are synchronized in the underlying inertial frame.

Centripetal potential, monopole potential, quadrupole, and higher potential terms conspire to give an equipotential in the rotating frame.

Earth-based Time Scale



$$L_G \equiv \frac{-\Phi_0}{c^2} \equiv 6.969290134 \times 10^{-10}.$$ (This number is now a defined quantity.)

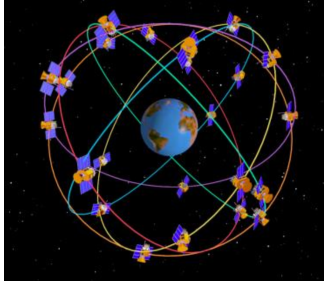
SI Second:

$$t \Rightarrow \left(1 - \frac{\Phi_0}{c^2}\right) t$$

$$(c\,d\tau)^2 = \left(1 + \frac{2(\Phi - \Phi_0)}{c^2}\right)(c\,dt)^2 - \left(1 - \frac{2\Phi}{c^2}\right)(dx^2 + dy^2 + dz^2)$$

(Basis for International Atomic Time, Universal Coordinated Time.)

Atomic Clock in a Satellite



$$\frac{1}{2}v^2 + \Phi = -\frac{GM}{2a}; \text{ a = semimajor axis}$$

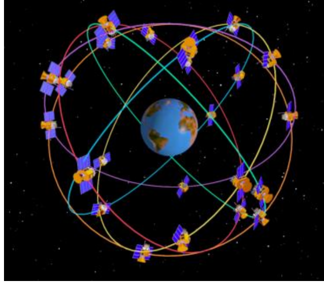
$$d\tau = \left(1 + \frac{2(\Phi - \Phi_0)}{c^2} - \frac{v^2}{c^2}\right)^{1/2} dt; \; dt = d\tau\left(1 - \frac{(\Phi - \Phi_0)}{c^2} + \frac{v^2}{2c^2}\right)$$

$$\Delta t_{\text{SV}} = \int_{\text{path}} d\tau\left[1 + \frac{3GM}{2ac^2} + \frac{\Phi_0}{c^2} - \frac{2GM}{c^2}\left(\frac{1}{a} - \frac{1}{r}\right)\right]$$

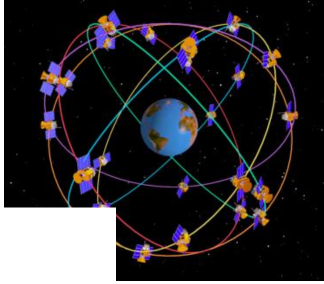
$$= \int_{\text{path}} d\tau[1 - 4.4647 \times 10^{-10}] \qquad -38.6\,\mu\text{s/day}$$

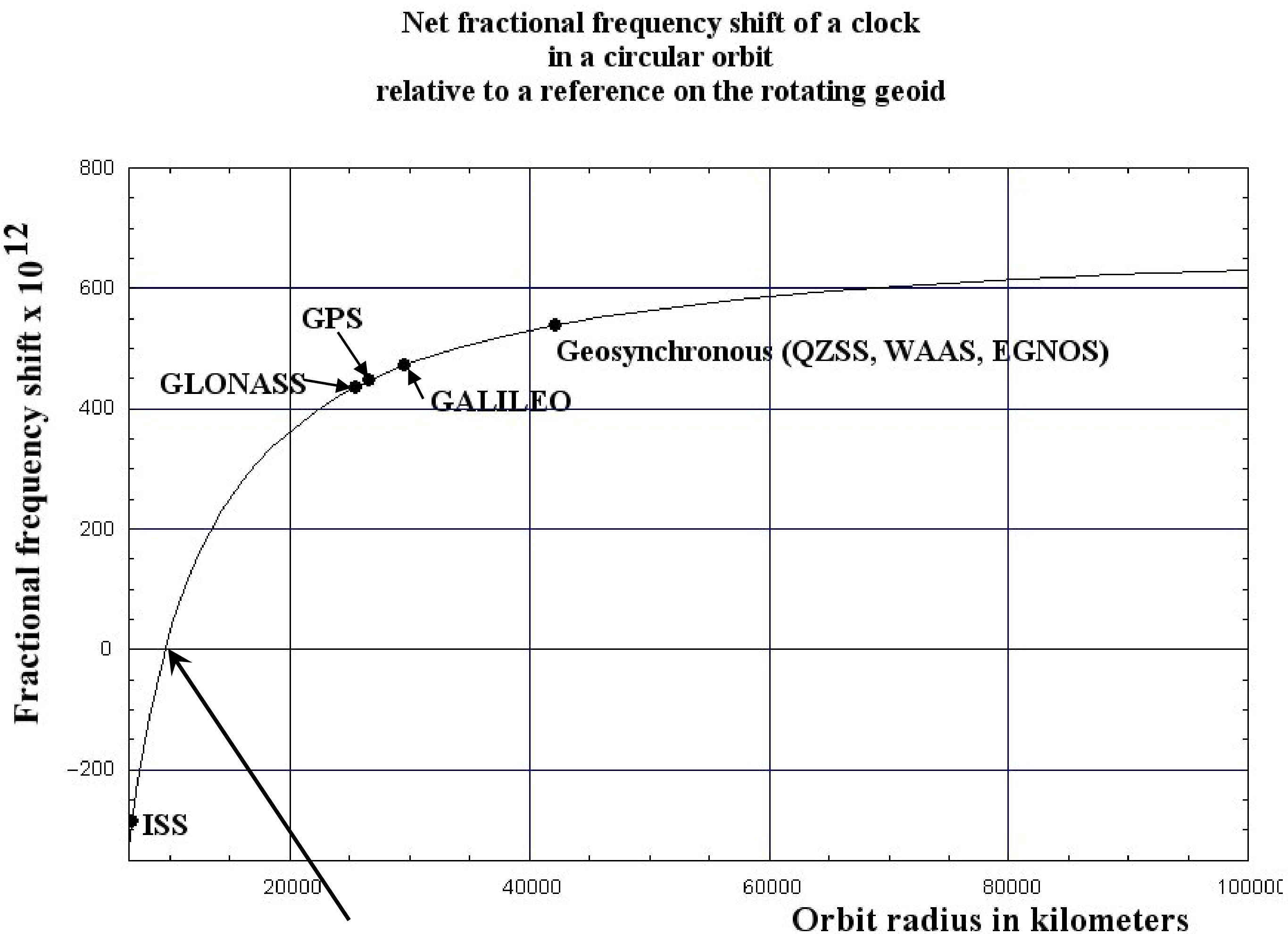
$$+\frac{2\sqrt{GM}}{c^2} e\sqrt{\frac{a}{\text{meter}}} \sin E \text{ (sec)}$$

Factory Frequency Offsets



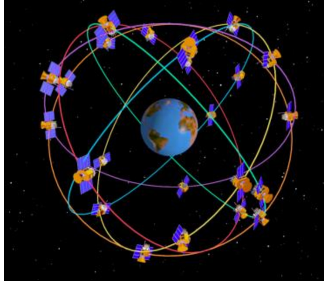
GNSS System	a (km)	$10^{12} \times \Delta f / f$
GLONASS	25509.64	-436.144
GPS	26562.76	-446.473
BEIDOU(MEO)	27910.20	-458.538
GALILEO	29601.31	-472.191
Geosynchronous	42164.17	-539.151





Frequency shifts cancel at this radius

Three Important Effects (GPS)



#1: Scale correction to satellite clock:

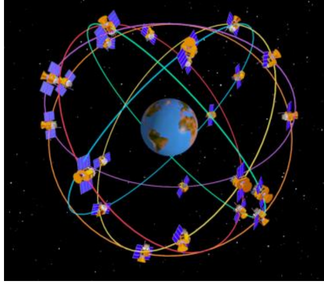
$$10.23000000000 \text{ MHz} \rightarrow 10.22999999543 \text{ MHz}$$

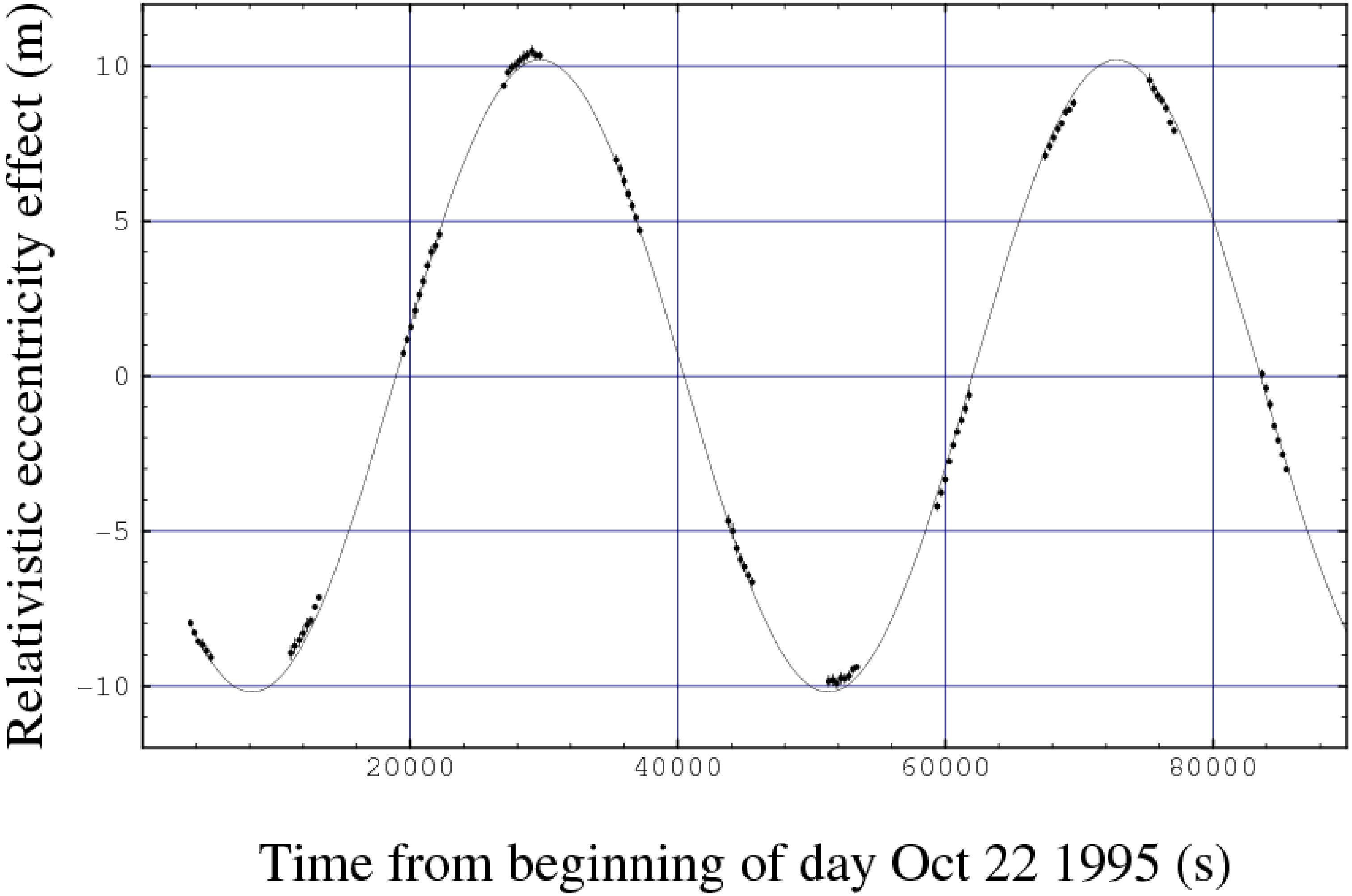
#2: Receiver must implement the eccentricity correction:

$$+4.4428\times 10^{-10} e\sqrt{\frac{a}{\text{meter}}}\sin E \ (\text{sec})$$

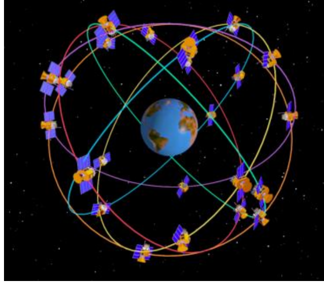
#3: User must account for time required for signal propagation (Sagnac effect) if relevant.

SV#13 eccentricity effect $e = 0.013$
(TOPEX receiver)





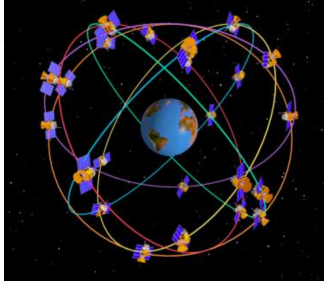
GALILEO Satellites in unintended orbits

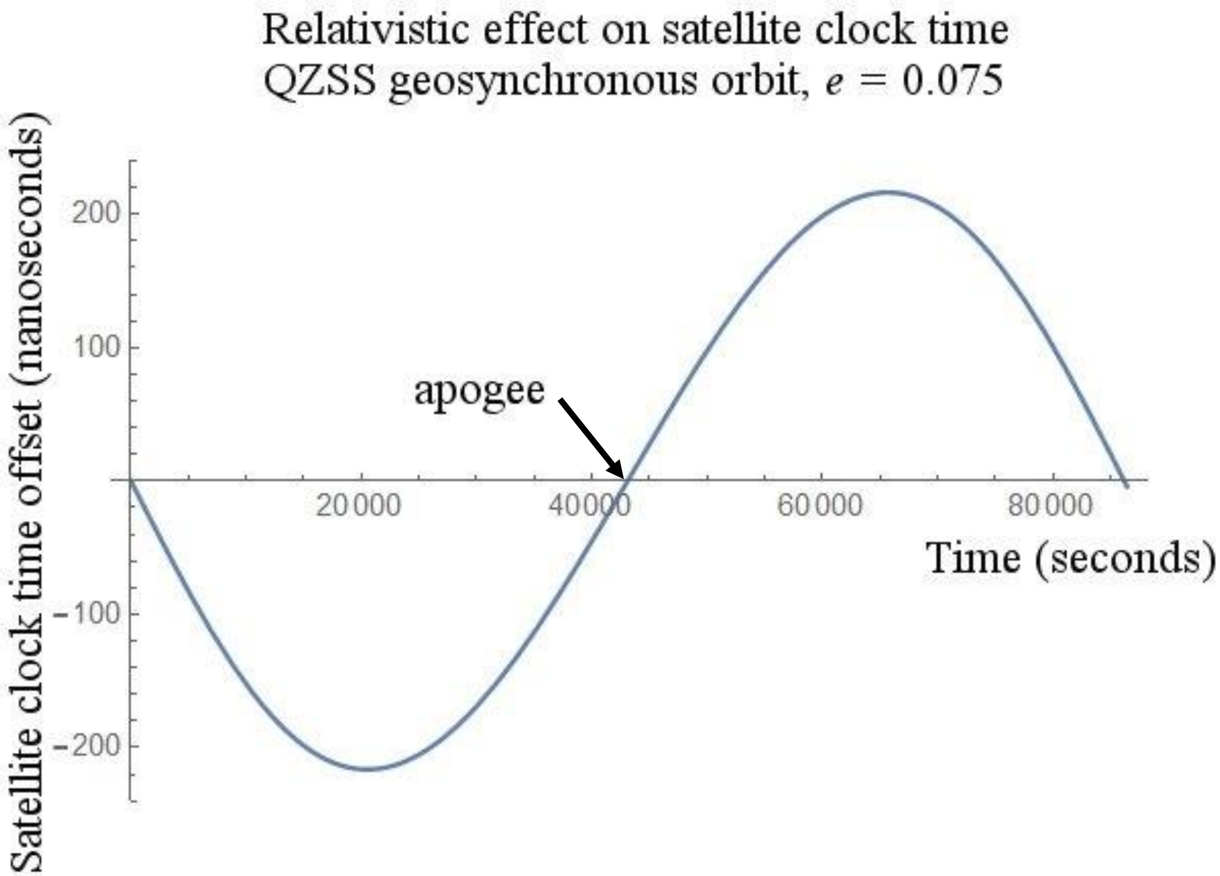


Extended Slots									
GSAT0201	18	Ext01	2014-08-22	27977.6	0.162	49.850	52.521	56.198	316.069
GSAT0202	14	Ext02	2014-08-22	27977.6	0.162	49.850	52.521	56.198	136.069

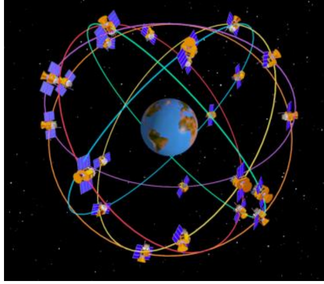
Normal radius of a GALILEO satellite: 29599.8 km
Eccentricity: 0

Quasi-Zenith Satellite System (Japan)





Frequency "Breaks" Due to Orbit Adjustments



$$\frac{\Delta f}{f} = -\frac{3GM}{2c^2 a} - \frac{\Phi_0}{c^2} + \frac{2GM}{c^2 a}\left[\frac{1}{r} - \frac{1}{a}\right].$$

If eccentricity is small,

$$\delta\left(\frac{\Delta f}{f}\right) = +\frac{3GM\,\delta a}{2c^2 a^2};$$

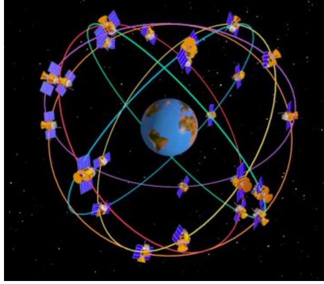
FOR SV#43:

Measured: -1.85×10^{-13}

Predicted: -1.77×10^{-13}

Now implemented.

Unmodeled Relativistic Effect: Oblateness



Effect of Earth's oblateness on satellite orbit:

$$\delta\Phi = \Phi_{J_2} = \frac{GMJ_2}{r^3}\left(\frac{3z^2 - r^2}{2r^2}\right);$$

Change in monopole potential:

$$\delta\left(-\frac{GM}{r}\right) = \frac{GMJ_2 a_1^2 \sin^2 I}{4a^3}\cos 2(\omega + f) + \ldots$$

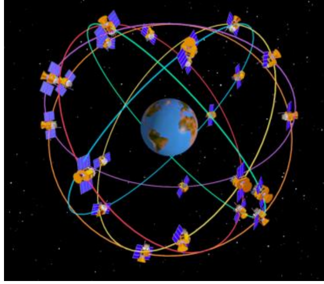
Change in kinetic energy:

$$\delta\left(\frac{v^2}{2}\right) = \frac{GMJ_2 a_1^2 \sin^2 I}{2a^3}\cos 2(\omega + f) + \ldots$$

Change in frequency:

$$\delta\left(\frac{\Delta f}{f}\right) = -\frac{GMJ_2 a_1^2 \sin^2 I}{a^3 c^2}\cos 2(\omega + f) + \ldots$$

A Coincidence?

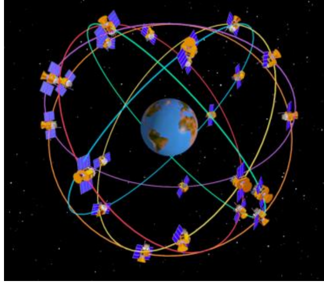


There are many terms in the perturbations arising from Earth’s oblateness with coefficient

$$\left(1-\frac{3}{2}(\sin I)^2\right)$$

For GPS, this is nearly zero. $(I = 55°)$

Shapiro delay SV to earth surface



$$\Delta t_{gravity} = -\frac{|r_2 - r_1| L_G}{c} + \frac{GM_e}{c^3} \log\left(\frac{r_1 + r_2 + |r_2 - r_1|}{r_1 + r_2 - |r_2 - r_1|}\right)$$

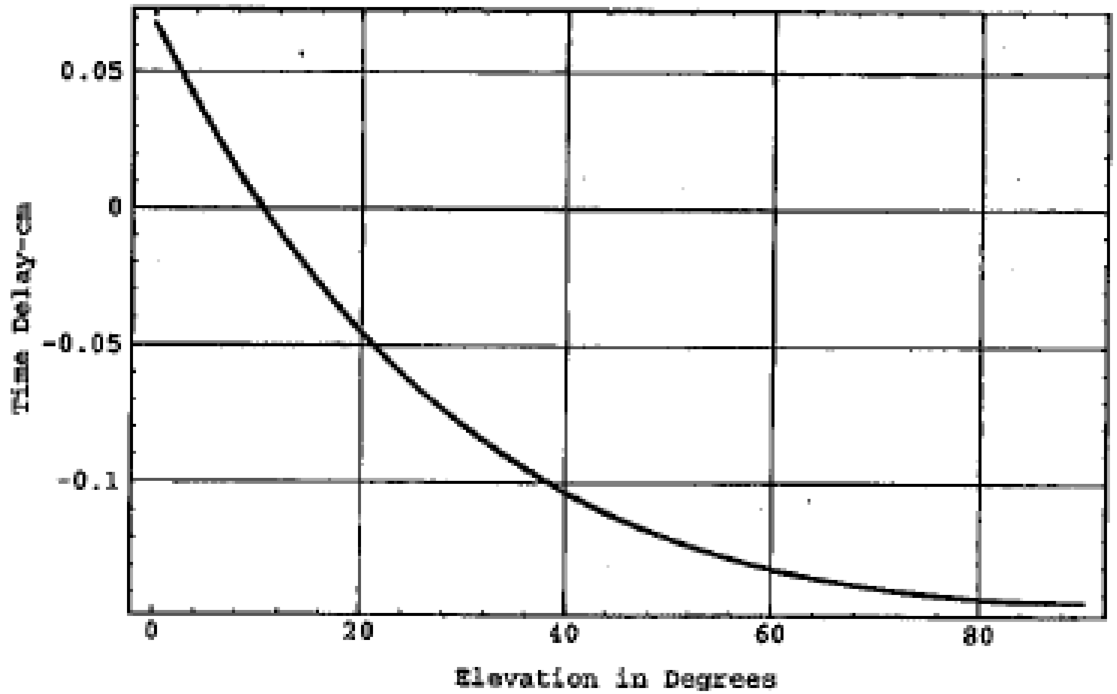
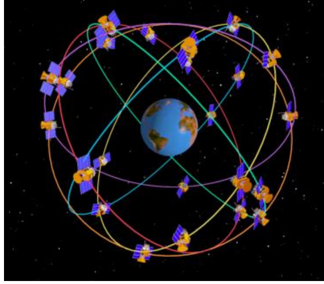


Fig. 31 Time delay vs elevation angle E for a GPS satellite-to-user link, including the time scale change for reference clocks on Earth's surface.

Spectrum of lunar tidal potential



Detailed calculation of the lunar tidal potential gives perturbations in terms of

$$\sum_i A_i \cos(n_i \omega_{sat} t + m_i \omega_{moon} t + \varphi_i)$$

$$n_i = -6, \ldots +8; \; m_i = -7, \ldots +8$$

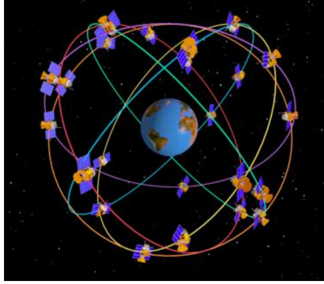
The coefficients are functions of the eccentricities and inclinations of the SV and the moon with respect to the equator. The phases are functions of the altitudes of perigee and the angles of the lines of nodes.

There are significant contributions from many frequencies in the neighborhood of 6 hours. (These correspond to $n_i = 2.$)

The short-period terms are sufficiently close together that they can beat against each other, reinforcing and cancelling. They can combine and have amplitudes that are estimated to be greater than about

$$2 \times 10^{-15}.$$

Unmodeled Relativistic Effects: Lunar and Solar Tides

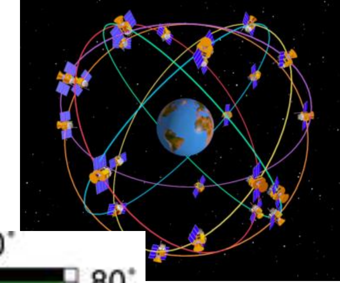


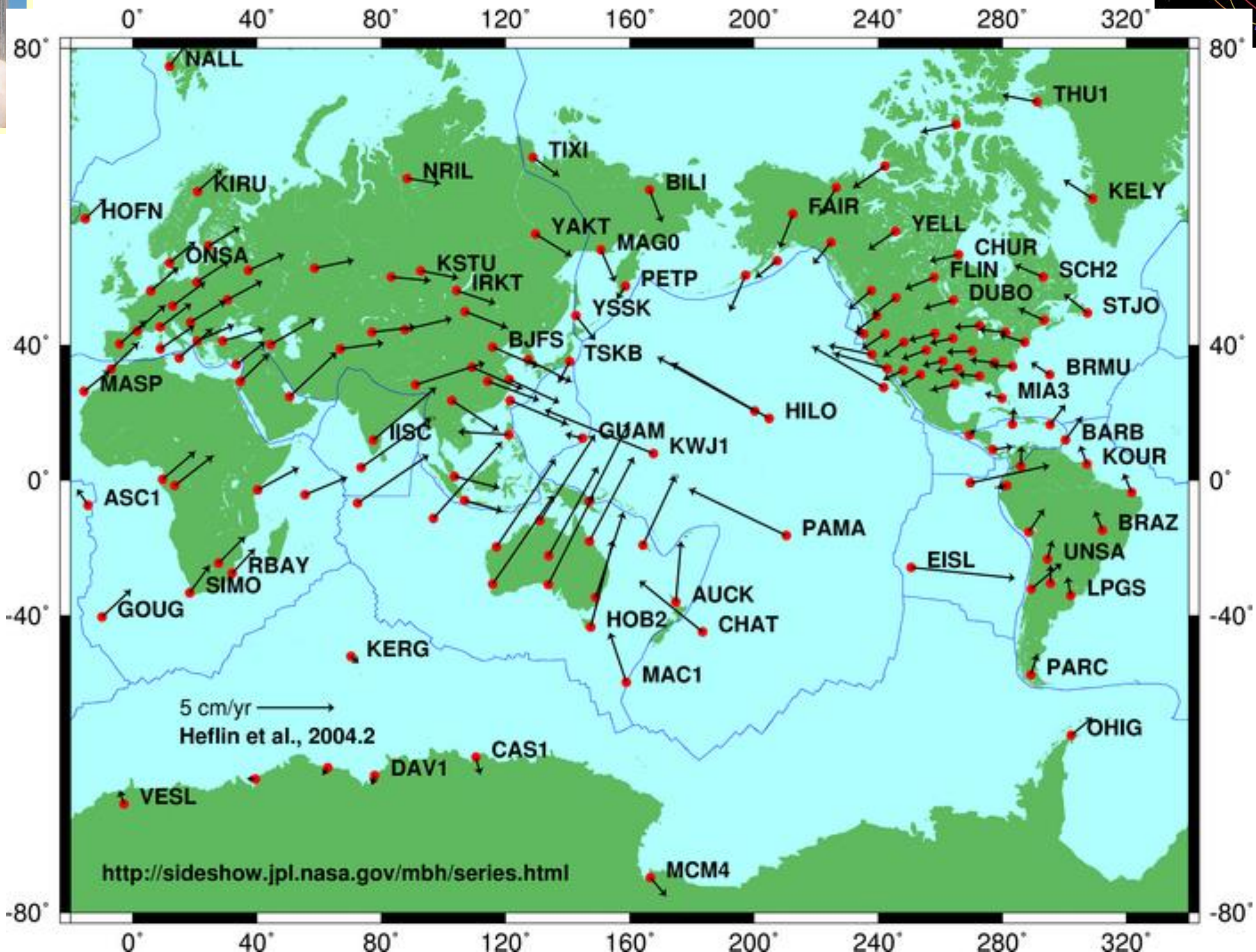
Lunar and solar tidal perturbations are estimated to affect the fractional frequency shifts of GPS SV clocks in a predictable way by about

$$3.7\times10^{-15}$$

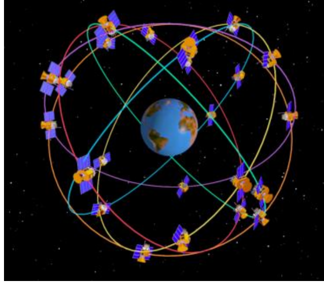
The principal periods with which this occurs are near 6 hours but there are many nearly equal frequencies.

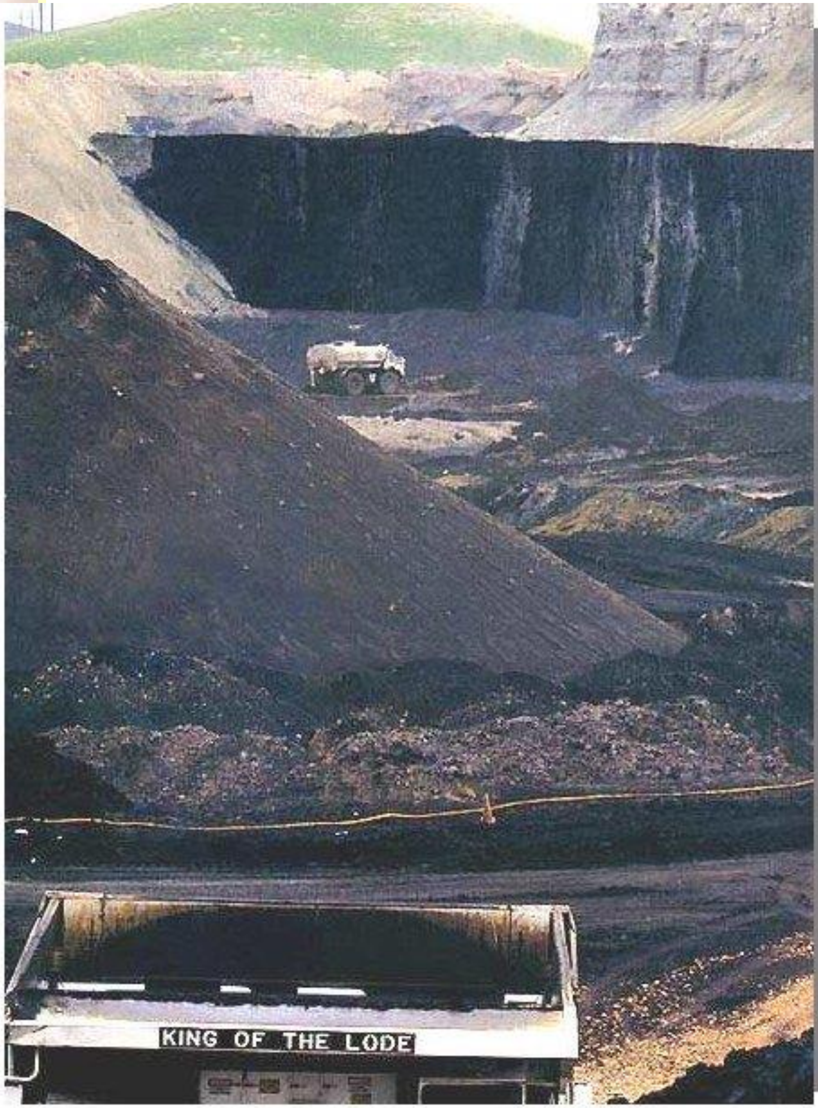
Surface Plate Velocities

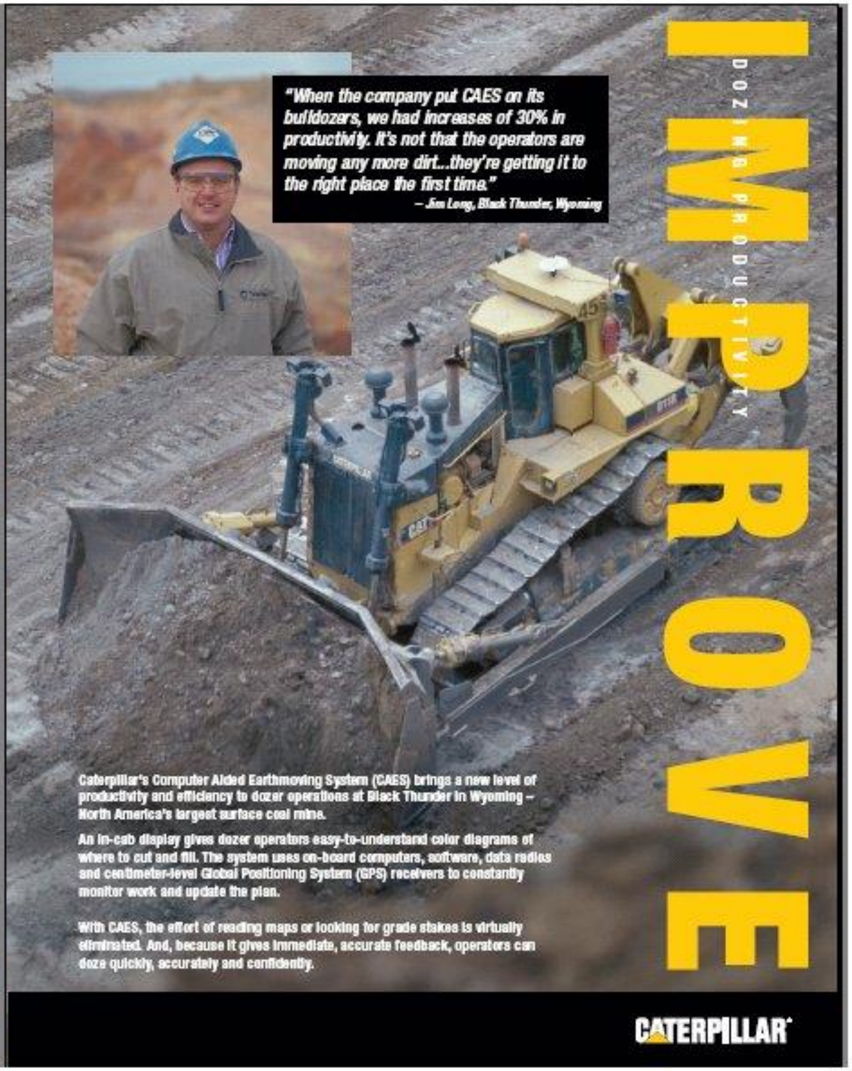




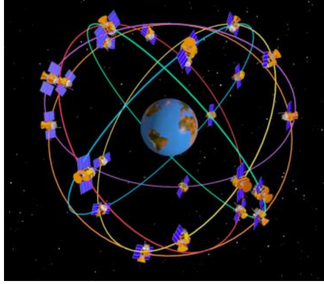
Control of Monster Machines



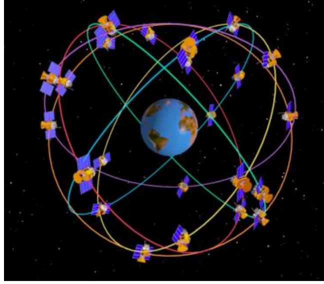




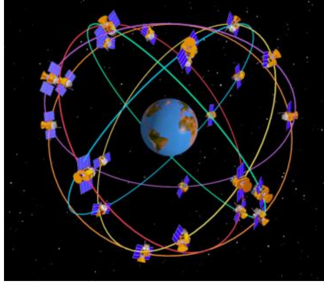
Autonomous Operation



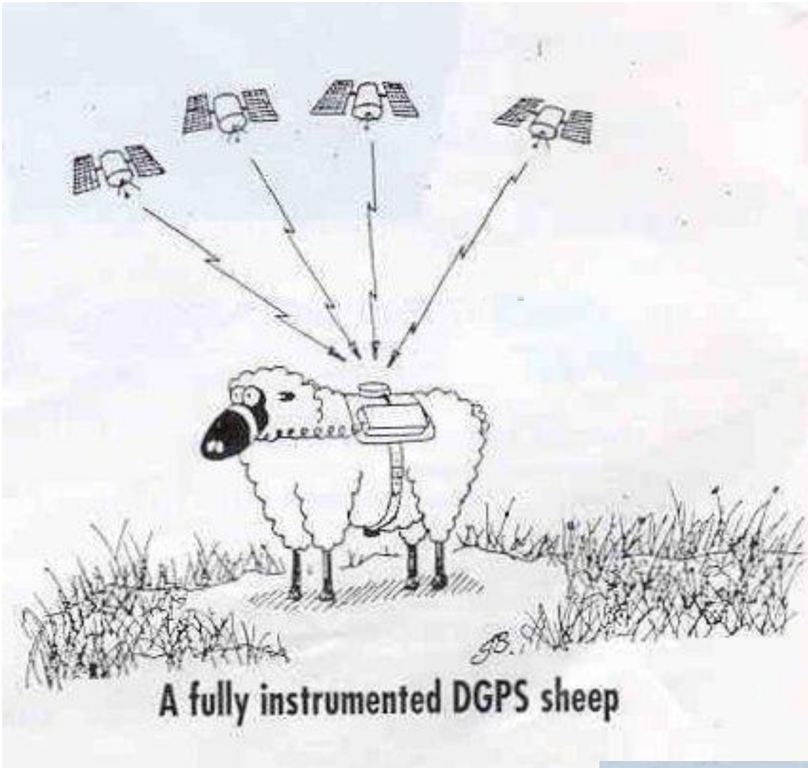
Precision Agriculture



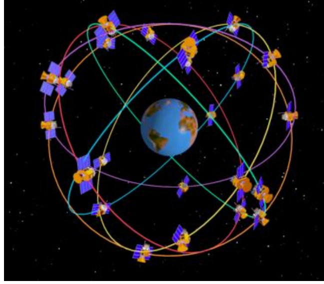
Surveying



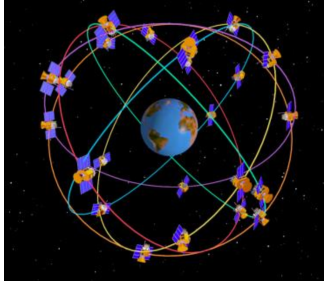
Finding boundary markers lost for a century.



Animal Tracking



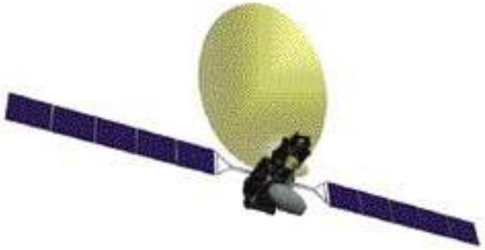
GNSS-other satellite navigation systems

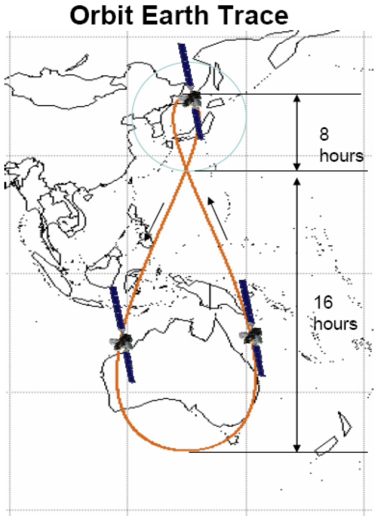


GLONASS-Russia
GALILEO-ESA
BEIDOU--China
IRNSS--INDIA
QZSS--JAPAN

AUGMENTATION SYSTEMS:
WAAS
EGNOS

QZSS

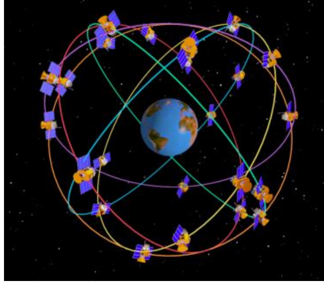


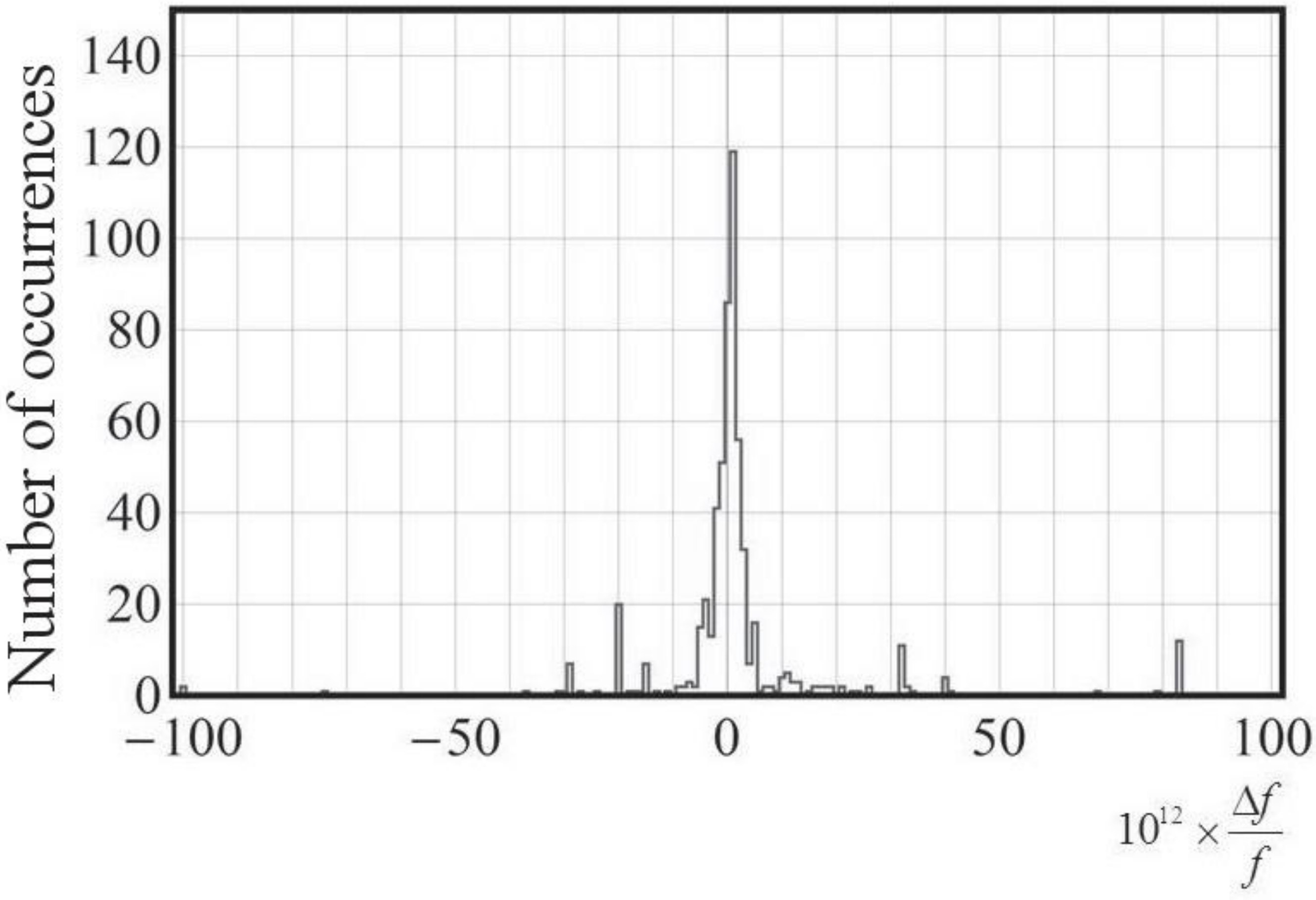


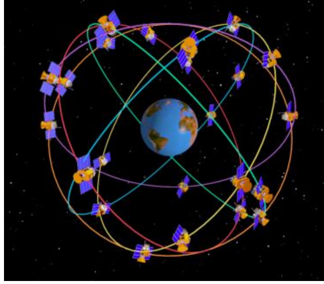
All use the same fundamental relativity concepts.

The GALILEO specs state “all relativity corrections are the responsibility of the user.” ????

Clock Coefficient a_0 of GPS Satellite clocks, 1992-2014







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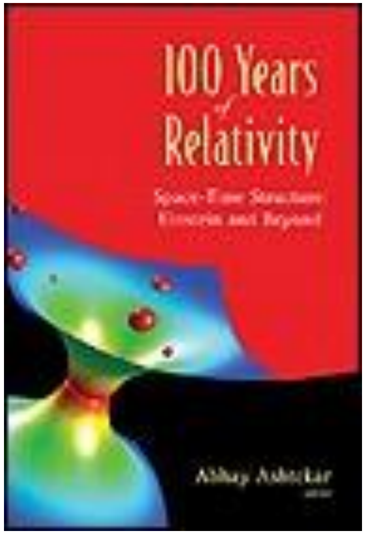
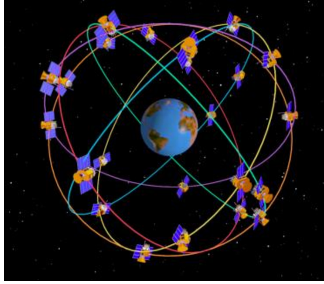
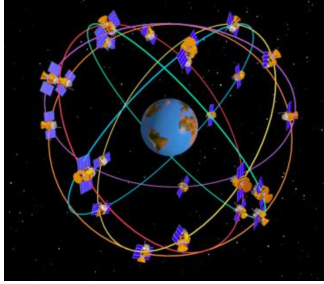
“General Relativity: The Most Beautiful of Theories,”
Rovelli, ed., de Gruyter, (2015) pp 165-188



END

GPS DEVELOPMENT KIT





Report of
SAMSO/NAVSTAR-GPS Relativity Seminar
National Bureau of Standards -Boulder, Colorado
8-9 March 1979

CONSENSUS:

1. To meet its design goals, GPS requires the consideration of relativistic effects. These effects are best described by the theory of General Relativity (which includes special relativity), and are fully deterministic. The theory is supported by the small amount of experimental data at an adequate level. Therefore the effects can and must be accounted for to the necessary levels of accuracy. The process of synchronization of clocks for GPS is best understood from a local, geocentric, nonrotating freely falling frame of reference.

2. The presentation by the staff of General Dynamics Electronics Division, as the contracting support organization, indicates that relativistic corrections are currently being implemented in GPS in a manner such that all significant corrections are included.

3. Appendix VI of Document CP-CS 304, Part I Code Ident 12436, 9 January 1978 entitled "Computer Program Development Specifications for the GPS Master Control Station Ephemeris Computer Programs" has significant errors in it and should be disregarded.

4. The relativistic correction due to the Sun (General Dynamics Electronics Division view graph 792K-039) showing an effect "of order 40 cm" is incorrectly calculated; however, this correction is not currently used and the effect is expected to be much smaller.

RECOMMENDATIONS:

1. General Dynamics Electronics Division should update their official documentation to correct errors such as those identified above.

2. The existing documentation is not suitable for communicating to the scientific and engineering community the nature of the relativistic corrections and the procedures by which they are implemented. More appropriate documentation should be prepared.

3. The designers of GPS user equipment in particular should have available adequate and proper documentation so that the appropriate relativistic corrections can be included.

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